

Partially Strategyproof Mechanisms for the Assignment Problem*

Timo Mennle[‡]

Sven Seuken[‡]

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Abstract

We study assignment mechanisms and propose a new way to trade off efficiency and strategyproofness. On the conceptual level, we make three key contributions. First, we propose a new paradigm we call *partial strategyproofness*, which is strategyproofness on a constrained subset of the utility space. Second, we introduce *hybrid mechanisms*, which are convex combinations of existing mechanisms. Third, we introduce *imperfect dominance*, a new concept to compare mechanisms by their efficiency properties. We show that hybrid mechanisms can *imperfectly dominate* their less efficient component. In contrast to *approximately strategyproof* mechanisms, our hybrids are also still “fully” strategyproof for agents whose utilities are bounded away from indifference, i.e., they are *partially strategyproof*. We present an algorithm which, for two admissible component mechanisms and a set of utility functions, computes the maximal mixing factor, such that the resulting hybrid is partially strategyproof. We instantiate our approach by mixing *Random Serial Dictatorship (RSD)* and *Probabilistic Serial (PS)*. The resulting hybrids imperfectly dominate *RSD* and are partially strategyproof. Finally, we present numerical results showing that the maximal mixing factor can be surprisingly high.

1. Introduction

In this paper, we study the *assignment problem*, also known as the *one-sided matching problem*. The first version of this problem was introduced by Hylland and Zeckhauser (1979), and has since attracted much attention of mechanism designers (e.g., Abdulkadiroğlu and Sönmez (1998); Bogomolnaia and Moulin (2001); Abdulkadiroğlu and Sönmez (2003b); Featherstone (2011)). The problem is to allocate a set of indivisible goods to a set of agents such that each agent receives one good. Agents have heterogeneous preferences over the goods, and these preferences are private information of the agents. Monetary transfers are not permitted, which makes this problem different from auctions and other settings with transferable utility. In practice, such problems often arise in situations that are of great importance to peoples’ lives. For example, students must be matched to schools, teachers to training programs, university students to courses, etc. (Niederle, Roth and Sönmez, 2008).

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[‡]Department of Informatics, University of Zurich, 8050 Zurich, Switzerland, {mennle, seuken}@ifi.uzh.ch.

As mechanism designers, we care specifically about *efficiency*, *strategyproofness*, and *fairness*. We want mechanisms that are efficient, i.e., that perform well with respect to some measure of social welfare. We want mechanisms that are strategyproof, i.e., where agents are best off reporting their true preferences to the mechanism. And we want mechanisms that satisfy some notion of fairness, such that conflicting agents’ interests are appropriately balanced. Zhou (1990) has shown that it is impossible to achieve the optimum on all three dimensions simultaneously, which makes the assignment problem an interesting mechanism design challenge. Obviously, trade-offs are necessary.

1.1. Trading off Efficiency for Strategyproofness

The best-known assignment mechanism is *Random Serial Dictatorship (RSD)* which first selects a random order of the agents, and then in this order, lets each agent select its most preferred good that is still available. *RSD* is strategyproof and also *ex-post efficient*, i.e., after all agents have been allocated their goods, no group of agents wants to exchange goods with each other. While this is certainly a desirable efficiency guarantee, it is also the weakest form of efficiency. For example, *ordinal efficiency* and *rank efficiency* are strict refinements of ex-post efficiency. However, the *RSD* mechanism does not guarantee either of these higher efficiency criteria, and it is well known that among deterministic mechanisms, the only ones that are strategyproof, neutral, nonbossy, and ex-post efficient are *dictatorships* (Hatfield, 2009). Allowing for non-deterministic mechanisms, it is a widely-held conjecture that *RSD* is the unique anonymous, ex-post efficient, and strategyproof mechanism. Thus, for a while, many researchers thought this is the end of the story, i.e., that we cannot do better than ex-post efficiency. But as Budish (2012) points out, these results do not mean that we should settle for ex-post efficiency because interesting trade-offs may be possible and worthwhile. In this paper, we show how to design mechanisms that have better efficiency properties than *RSD*, but are still strategyproof for agents whose utilities are bounded in a certain way.

In the hierarchy of efficiency concepts, *ordinal efficiency* is the the first refinement of ex-post efficiency: it requires that we cannot find a set of agents who can trade probability shares such that the resulting allocation ordinally dominates the original allocation for those agents. Bogomolnaia and Moulin (2001) introduced the *Probabilistic Serial (PS)* mechanism and showed that it is *ordinally efficient*. However, they also showed that *PS* is not strategyproof in general, and that there exists no mechanism that is ordinally efficient, anonymous, and strategyproof in all settings. Kojima and Manea (2010) showed that for large assignment problems with a fixed number of goods, *PS* is strategyproof if the number of copies of each good is sufficiently large. In contrast, in this paper we design mechanisms that also provide strategyproofness in *small settings*, but only for agents with slightly constrained utility functions.

In recent years, there has been a lot of work explicitly tackling the trade-off between efficiency and strategyproofness in mechanism design. However, the vast majority of it has aimed for *approximate* notions of strategyproofness (e.g., Budish (2011); Carroll (2012)). Typically, *approximate strategyproofness* means relaxing “full” strategyproofness, and instead bounding the incentives for deviation from a truthful report for any agent. In contrast, our mechanisms remain “fully” strategyproof on the constrained subset of utilities.

1.2. Partial Strategyproofness, Hybrid Mechanisms, and Imperfect Dominance

The main contribution of this paper is a new paradigm we call *partial strategyproofness*, which describes mechanisms that are strategyproof, but only for a constrained set of the utility space. The main constraint we place on agents’ utilities is a *bound on how close to indifferent* agents can be. Intuitively, those agents at the “boundary,” i.e., who are indifferent or almost indifferent between two goods, are the first to manipulate a mechanism, should it not be strategyproof. Thus, explicitly excluding these agents helps us to design mechanisms that are strategyproof for the remaining agents with less “extreme” utilities. Additionally, we bound the maximum relative utility difference between an agent’s least and most preferred goods, thereby limiting how much an agent can benefit from any manipulation. Formally, for a setting with m goods, we introduce the set $URB(r, B, m)$ to denote the space of *uniformly relatively bounded* utility functions: r denotes the relative bound between utilities for any two goods that are consecutive in an agent’s preference ordering, and B denotes the relative bound between an agent’s least and most preferred good. Note that we operate with and carefully reason about agents’ cardinal utilities. However, throughout this paper we will only consider mechanisms that elicit agents’ ordinal preferences, i.e., rankings over goods.

This relaxation to partial strategyproofness enlarges the mechanism design space and enables new trade-offs between efficiency and (partial) strategyproofness. In particular, it allows us to analyze the incentive properties of *hybrid mechanisms*, which are convex combinations of existing mechanisms. Given a strategyproof mechanism f with low efficiency, and a manipulable mechanism g with higher efficiency, the goal is to design a hybrid $h_\beta(f, g)$ with intermediate efficiency. The parameter β describes the mixing factor between f and g . Of course, these new hybrids will generally not be strategyproof. However, we show how to design hybrids that are partially strategyproof, i.e., strategyproof on a specific set $URB(r, B, m)$.

To analyze the efficiency of hybrid mechanisms, we introduce a new concept we call *imperfect dominance*, which is a natural extension of the canonical notions of ordinal and rank dominance to mechanisms. Formally, we write $h \geq_{IOD} f$ to say that a mechanism h *imperfectly ordinaly dominates* a mechanism f . Intuitively, this means that whenever the outcomes of h and f are comparable with respect to ordinal dominance, then the outcome of h ordinaly dominates the outcome of f . Imperfect dominance is a useful concept to compare the efficiency properties of mechanisms, because it allows us to make meaningful comparisons between two mechanism, even if not all of the allocations they produce are comparable. This relaxation is essential for the analysis of hybrids, since dominance of the allocations cannot always be guaranteed.

1.3. Overview of Contributions

The most tangible result of this paper is a hybrid mechanism that mixes *RSD* and *PS*, is strategyproof on $URB(r, B, m)$, and imperfectly ordinaly dominates *RSD*. But before we get to this result, we go through a number of interesting intermediate steps.

We first introduce two new concepts: *partial strategyproofness* of mechanisms and *uniform relatively bounded utility functions* (Def. 1 & 5). We also introduce *imperfect dominance*, a new notion of comparing mechanisms by their efficiency and argue why it is a useful extension of dominance from allocations to mechanisms (Prop. 6 & 7, Def. 10). We now present an overview of our main technical results. See Figure 1 for how they are connected.

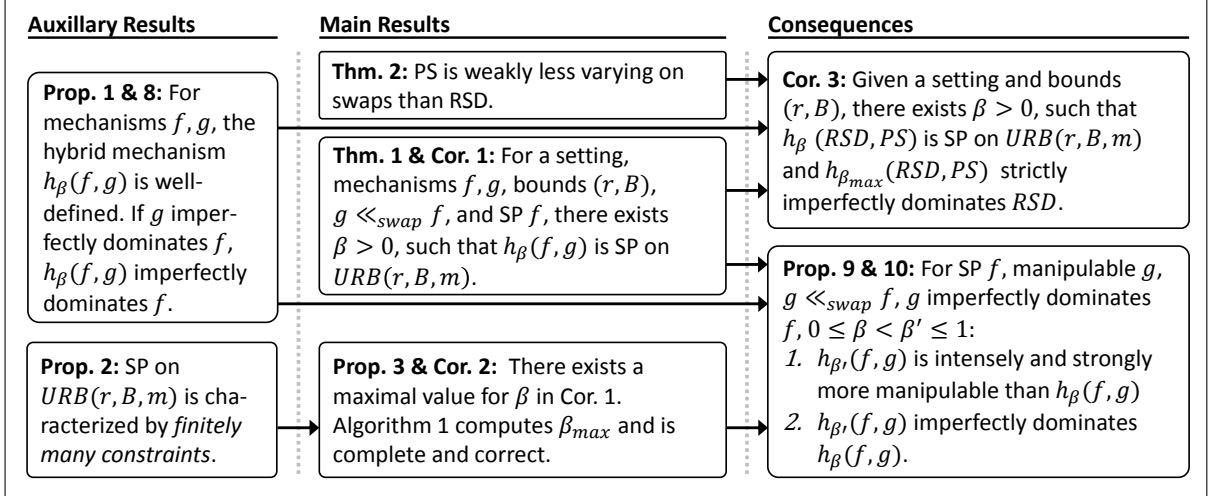


Figure 1: Overview of formal results and proof structure (SP stands for *strategyproofness*).

1. We show that for any two mechanisms f and g , the hybrid mechanism $h_\beta(f, g)$ is a well-defined mechanism (Prop. 1) and can imperfectly dominate the less efficient component (Prop. 8). We introduce the new concept $g \ll_{\text{swap}} f$, which requires that if g changes the allocation due to some change of report (through a swap), then so does f . In Thm. 1 & Cor. 1, we show that if a pair of mechanisms satisfies this property, we can use them to construct non-trivial hybrids that are strategyproof on $URB(r, B, m)$.
2. Our second set of contributions relates to the computability of the maximal mixing factor β . In Prop. 2, we show that strategyproofness on $URB(r, B, m)$ can be verified by checking a *finite* number of constraints. In Prop. 3 & Cor. 2, we show that given a setting (i.e., agents, goods, and capacities), mechanisms f, g , and bounds (r, B) , there exists a maximal value β_{max} for β . Then we present Algorithm 1 which computes this β_{max} , and we show that the algorithm is complete and correct.
3. In Props. 9 & 10, we show that our parametrization of hybrid mechanisms leads to a natural *hierarchy of efficiency and manipulability*: increasing β leads to a strictly imperfectly dominant hybrid mechanism which is also strictly more manipulable.
4. Finally, we instantiate our general results using the mechanisms RSD and PS . In Thm. 2, we show that $PS \ll_{\text{swap}} RSD$. In Cor. 3, we use this to show that RSD and PS can be mixed such that the hybrid $h_\beta(RSD, PS)$ is strategyproof on $URB(r, B, m)$ and strictly imperfectly ordinally dominates RSD . Finally, we present numerical results illustrating the range of β_{max} for different sets of bounds when mixing RSD and PS , and we find that β_{max} is surprisingly large.

2. Related Work

While the seminal paper on assignment mechanisms by [Hylland and Zeckhauser \(1979\)](#) proposed a mechanism that elicits agents’ cardinal utilities, this approach has proven problematic because it is difficult if not impossible to elicit cardinal utilities in settings without money. For this reason, recent work on matching has focused on *ordinal mechanisms*, where agents submit ordinal preference reports, i.e., rankings over all goods (see [Abdulkadiroğlu, Pathak and Roth \(2005\)](#); [Calsamiglia and Miralles \(2012\)](#)) for examples, or [Carroll \(2011b\)](#) for a systematic argument). Throughout this paper, we only consider ordinal mechanisms. However, similar to the analysis technique used in [Carroll \(2012\)](#), in all of our proofs we will carefully reason about agents’ cardinal utilities that are consistent with their ordinal preferences. This style of analysis is necessary because our results hinge on utility functions to be bounded away from indifference, which in turn requires modeling cardinal utility functions.

In recent years, many results have further characterized the space of strategyproof assignment mechanisms. For example, [Hatfield \(2009\)](#) shows that the only deterministic, strategyproof, ex-post efficient, nonbossy, and neutral mechanisms are serial dictatorships. For non-deterministic mechanisms, [Abdulkadiroğlu and Sönmez \(1998\)](#) showed that *RSD* is equivalent to the Core from Random Endowments mechanism for house allocation, if agents’ initial houses are drawn uniformly at random. However, it is still an open conjecture whether *RSD* is the unique mechanism that is anonymous, ex-post efficient, and strategyproof.

The research community has also further characterized the space of ordinally efficient mechanisms. The original *PS* mechanism introduced by [Bogomolnaia and Moulin \(2001\)](#) was only defined for strict preferences. [Katta and Sethuraman \(2006\)](#) introduced an extension of the *PS* mechanism that allows agents to be indifferent between goods, and [Heo and Yilmaz \(2011\)](#) further characterized their solution. Recently, [Hashimoto et al. \(2013\)](#) have shown that the unique mechanism that is ordinally fair (and non-wasteful) is *PS* with uniform eating speeds. [Bogomolnaia and Moulin \(2001\)](#) had already shown that *PS* is not strategyproof, but [Kesten and Ekici \(2012\)](#) recently found that the Nash equilibria of the corresponding revelation game may result in ordinally inefficient outcomes. While the incentive properties of *PS* may be bad for small games, they get better for larger games. As mentioned in the introduction, [Kojima and Manea \(2010\)](#) have shown that for a fixed number of goods, *PS* is strategyproof if the number of copies of each good is sufficiently large.

It turns out that good incentives in large games are not a property reserved for *PS*. Recently, [Azevedo and Budish \(2012\)](#) proposed a new desideratum for mechanism design called *Strategyproof in the Large* (SP-L), which formalizes the intuition that as the number of agents in the market gets large, the incentives for individual agents to misreport their preferences should vanish in the limit. [Azevedo and Budish \(2012\)](#) show that this property is satisfied for many well-known mechanisms, including *PS*. However, it once again sheds light on the fact that incentive guarantees are hard to control in small settings, which is where our approach can shine.

While ex-post efficiency and ordinal efficiency are the two most well-studied efficiency concepts for assignment mechanisms, many mechanisms are used in practice that satisfy neither efficiency concept. [Featherstone \(2011\)](#) observed that many mechanisms in practice aim to

maximize *rank efficiency* which is a further refinement of ordinal efficiency. However, no rank efficient mechanisms can be strategyproof in general. Other popular mechanisms, like the Boston Mechanism (see [Ergin and Sonmez \(2006\)](#)), are highly manipulable but are nevertheless in frequent use. [Budish and Cantillon \(2012\)](#) show practical evidence from combinatorial course allocation, suggesting that using a non-strategyproof mechanism may lead to higher social welfare than using an ex-post efficient mechanism such as *RSD*. This illustrates that ex-post efficiency is a relatively weak criterion, at least for some settings.

Given that strategyproofness is such a strong restriction in the assignment problem, many researchers have tried to relax it, using various notions of *approximate strategyproofness*. For example, [Carroll \(2011a\)](#) takes this approach and quantifies the incentives to manipulate for voting rules. [Budish \(2011\)](#) takes a similar approach in the domain of combinatorial assignment problems. Finally, [Dütting et al. \(2012\)](#) use a machine learning approach to design mechanisms with “good” incentive properties. Instead of requiring strategyproofness, they minimize the agents’ ex-post regret, i.e., the utility increase an agent could gain from manipulating. In contrast to all of these existing approaches, in our work we do not directly trade off efficiency for strategyproofness, because we are not losing strategyproofness for *all* agents. Instead, we reduce the space of utility functions for which strategyproofness is required, thus making a different kind of trade-off. We are not aware of any existing work on assignment mechanisms aiming at such a quantitative (rather than structural) restriction.

Given that mechanism designers often face the necessity to make trade-offs between efficiency and strategyproofness, it is natural that we desire a way to quantify or compare the manipulability of different mechanisms. [Lubin and Parkes \(2012\)](#) provide a survey of different ways in which a mechanism designer can relax strategyproofness and potentially quantify the degree of manipulability. Recently, [Pathak and Sönmez \(2013\)](#) introduced a general framework for comparing mechanisms by their vulnerability to manipulations. We apply their *hierarchy of manipulability* concept to our framework and show that a hybrid mechanism $h_\beta(f, g)$ becomes what they call “intensely and strongly more manipulable” as the mixing factor β increases. This result establishes a nice connection between their most demanding comparison and our mechanism design approach.

3. The Model

Suppose that m different goods must be allocated to n agents, and there exist q_j copies of each good j . A *setting* (N, M, \mathbf{q}) consists of the set of *agents* N , the set of *goods* M , and the vector of *capacities* $\mathbf{q} = (q_1, \dots, q_m)$ ¹. Note that the setting does not specify the agents’ preferences, merely their names. An outside option can be modeled by including an additional dummy good o , which is available in unlimited supply (or $q_o \geq n$). Thus, w.l.o.g., we can assume that there are enough goods for all agents, i.e., $\sum_{j \in M} q_j \geq n$.

¹To indicate a vector, we sometimes use bold notation, such as in \mathbf{q} .

3.1. Allocations

An *allocation* X is an $n \times m$ -matrix of values between 0 and 1 that satisfies

1. $\sum_{j \in M} x_{i,j} = 1$ for all $i \in N$, and
2. $\sum_{i \in N} x_{i,j} \leq q_j$ for all $j \in M$.

The elements $x_{i,j}$ are interpreted as the *probability that agent i receives good j* . 1. means that every agent has probability 1 for receiving some good. 2. ensures that no good is allocated beyond its capacity. An allocation is *deterministic* if all entries of the matrix are either 0 or 1, otherwise it is *probabilistic* or *non-deterministic*. Let \mathcal{X} be the set of all allocations, and let $\mathcal{D} \subset \mathcal{X}$ be the set of all deterministic allocations.

Every allocation can be represented as a convex combination of deterministic allocations. This is a result from Budish et al. (2013), which extends the famous Birkhoff-Von Neumann Theorem (see von Neumann (1953)) to settings with multiple copies of each good. The convex combination can be interpreted as a *lottery*, where each deterministic allocation is selected with probability equal to its coefficient in the convex combination. In such a lottery agent i receives good j with probability $x_{i,j}$.²

3.2. Ordinal Preferences and Cardinal Utilities

For an agent i , the relation \succ_i denotes its (*ordinal*) *preference order* over goods: $a \succ_i b$ if for two goods a and b the agent prefers good a over good b . Agent i weakly prefers good a over good b if either $a \succ_i b$ or $a = b$, which we denote by $a \succeq_i b$. Thus, throughout this paper we explicitly exclude indifference between two goods (see Section 4.2 for details). Note that we often use $>$ instead of \succ_i when we are not referring to a specific agent i . The collection of all agents' preferences $\succ = (\succ_1, \dots, \succ_n)$ is called the *preference profile*.

A preference order over goods induces a weak preference order over deterministic allocations for each agent: suppose agent i receives good a under X and good b under Y , i.e., $x_{i,a} = 1$ and $y_{i,b} = 1$, then $X \succeq_i Y$ if $a \succeq_i b$. If agent i likes the good it receives under X at least as much as the good it receives under Y , then we write $X \succeq_i Y$.

From a single agent's perspective, non-deterministic allocations are not always comparable in this way. Fortunately, von Neumann and Morgenstern (1944) have shown that assuming a set of mild axioms, agents behave *as if* maximizing expected utility with respect to *some* cardinal utility function, called *Von Neuman-Morgenstern (VNM) utility*. We assume that all agents have VNM utility functions (or just *utilities*), such that the utility u is *consistent with* the agent's preference order \succ , i.e., $u(a) > u(b)$ if $a \succ b$.

We adopt the geometric interpretation of utilities and types from Carroll (2012): a utility u is interpreted as an m -dimensional vector of positive real values. The space of all such vectors

²The lottery representation (i.e., which deterministic allocations are combined to obtain a non-deterministic allocation) is not unique, and the choice of lottery can have welfare and incentive implications, as discussed in Example 2 in Abdulkadiroğlu and Sönmez (2003a). In this paper, we do not consider the lottery implementation, but only compare allocations with respect to the agents' ordinal preferences and their expected utility. These measures are invariant to the choice of lottery.

is $U = (\mathbb{R}^+)^m$, called *utility space*. For a given preference order $>$, a *type*

$$t = t_{>} = \{u \in U : u \text{ consistent with } >\}$$

is the set of all utilities consistent with $>$. We say an agent i is of (or has, or is in) *type* t_i , if its utility u_i lies within type t_i . The *type space* T is the set of all types, and for all n agents the collection of types $\mathbf{t} = (t_1, \dots, t_n)$ is the *type profile*.

We use the same notion of similarity on types as used by Carroll (2012): the *neighborhood* N_t of some type $t \in T$ is the set of all types for which the preference orders of agents in t and agents in $t' \in N_t$ differ by just a swap of two adjacent goods in the rankings. Formally, for type t and $t' \in N_t$ there exist goods x, y such that agents in t have preference order

$$>: a_1 > \dots > a_{m_a} > x > y > b_1 > \dots > b_{m_b},$$

and agents in t' have preference order

$$>': a_1 >' \dots >' a_{m_a} >' y >' x >' b_1 >' \dots >' b_{m_b}.$$

3.3. Assignment Mechanisms

An *assignment mechanism* is a mapping f that determines an allocation of the goods based on agents' type reports. Formally, it is a mapping $f : T^n \rightarrow \mathcal{X}$ from the *space of type profiles* to the space of allocations. f is *deterministic* if $f : T^n \rightarrow \mathcal{D}$, i.e., the resulting allocation only assigns discrete (non-fractional) goods.

Because types are private information of the agents, when asked to report their type, they may lie about it. When we need to explicitly distinguish between agent i 's true and reported type, we write t_i for the true type, and t'_i for the misreport. By $t_{-i} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ we denote the collection of types of all agents excluding i .

We let $f(t_i, t_{-i})_i$ denote the allocation for agent i given reports (t_i, t_{-i}) , i.e., the i th row of the allocation matrix. For a good j , let $f(t_i, t_{-i})_{i,j}$ denote the respective entry of this row. Making use of the geometric set-up, the expected utility for agent i can be expressed as the dot product of the vectors u_i and $(f(t_i, t_{-i}))_i$, i.e.,

$$(E_{f(t_i, t_{-i})}[u])_i = \langle u_i, f(t_i, t_{-i})_i \rangle = \sum_{j \in M} u_{i,j} \cdot f(t_i, t_{-i})_{i,j},$$

where $u_i \in t_i$ denote the utility and type of agent i .

We now define properties a mechanism f can have.

Definition 1 (Partial Strategyproofness). *For a fixed setting (N, M, \mathbf{q}) , an agent i of type t_i and with utility $u \in t_i$, f is u -strategyproof in the setting (N, M, \mathbf{q}) if in this setting it is a dominant strategy for i to report truthfully, i.e., for any $t'_i \in T$, and $t_{-i} \in T^{n-1}$ we have*

$$\langle u, f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i \rangle \geq 0. \quad (1)$$

This means that i maximizes its expected utility by reporting truthfully.

For a set A of utility functions, f is strategyproof on A in the setting (N, M, \mathbf{q}) if it is u -strategyproof in the setting (N, M, \mathbf{q}) for all $u \in A$. If the set A and the setting are clear from the context, we say that f is partially strategyproof.

Definition 2 (Strategyproofness). f is strategyproof (SP) if for any set of utility functions in any setting f is partially strategyproof.

Definition 3 (Fairness: Non-bossiness, Anonymity, Equal Treatment of Equals). f is non-bossy if no agent can change the overall outcome without changing its own allocation, i.e., for all agents i of type t_i , types $t'_i \in T$, and $t_{-i} \in T^{n-1}$,

$$f(t_i, t_{-i}) \neq f(t'_i, t_{-i}) \Rightarrow f(t_i, t_{-i})_i \neq f(t'_i, t_{-i})_i.$$

f is anonymous if the order of the agents does not matter, i.e., for all agents i and any permutation $\pi : N \rightarrow N$ of the agents, we have

$$f(t_1, \dots, t_n)_i = f(t_{\pi(1)}, \dots, t_{\pi(n)})_{\pi(i)}.$$

Anonymity implies that all agents of the same type must receive the same (non-deterministic) allocation. This is also called equal treatment of equals.

In proofs, we sometimes abbreviate $f(t, t_{-i})_i$ as $f(t)$ if the collection of other agents' preferences t_{-i} and the particular agent i are clear from the context.

3.4. Efficiency Concepts

We now define different efficiency concepts for allocations. A mechanism f satisfies a particular efficiency concept if it produces allocations that satisfy the respective efficiency concepts with respect to the reported preferences.

Ex-post Efficiency. A deterministic allocation $D \in \mathcal{D}$ is *ex-post efficient* if no group of agents could exchange their assigned goods such that all agents weakly prefer their new assignment and at least one agent strictly prefers the new assignment. Formally, there exists no $D' \in \mathcal{D}$ such that for all agents i we have $D' \succeq_i D$, and $D' \succ_{i'} D$ for at least one agent i' . A non-deterministic allocation is *ex-post efficient* if it has a lottery representation consisting only of ex-post efficient deterministic allocations.

Ordinal Efficiency. To define ordinal efficiency we first need to introduce the notion of stochastic dominance. Let X and Y be allocations. For agent i , X_i *first order-stochastically dominates* Y_i at \succ_i if for all goods j

$$\sum_{j' \succeq_i j} x_{i,j'} \geq \sum_{j' \succeq_i j} y_{i,j'}. \quad (2)$$

This means that for any good j , agent i 's probability of receiving a good that it likes at least as much as j is no smaller under X than under Y .

X *ordinally dominates* Y (at \succ) if for all agents i , X_i first order-stochastically dominates Y_i at \succ_i , and we write $X \succeq_{OD} Y$. If there exists at least one agent i and one good j for which inequality (2) is strict, then we say that X *strictly ordinally dominates* Y (at \succ) and write $X \succ_{OD} Y$. Finally, an allocation X is *ordinally efficient* (at \succ) if it is not strictly ordinally dominated by any other allocation (at \succ). Intuitively, this means that probability

shares cannot be reassigned amongst the agents such that all agents weakly prefer the new allocation and at least one agent is strictly better off in the sense of first order-stochastic dominance. Note that on deterministic allocations, ex-post efficiency and ordinal efficiency coincide. Example 1 (borrowed from Section 2 in [Bogomolnaia and Moulin \(2001\)](#)) shows that for non-deterministic allocations, ordinal efficiency is a true refinement of ex-post efficiency.

Example 1. Suppose $N = \{1, 2, 3, 4\}$, $M = \{a, b, c, d\}$, $q_j = 1$ for all goods j . Agents' preferences are

$$\begin{aligned} >_1, >_2 & : a > b > c > d, \\ >_3, >_4 & : b > a > d > c. \end{aligned}$$

Consider the allocations

$$X = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}, Y = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

We see that each agent strictly prefers allocation X , e.g., agent 1 traded all its shares of b for shares of a and all shares of d for shares of c with either agent 3 or 4. Also note that both allocations are ex-post efficient, but only X is ordinally efficient.

Rank Efficiency. The rank of a good j for some agent i is the number of goods that i weakly prefers to j , denoted by $\text{rank}_i(j)$. An allocation X rank dominates another allocation Y if for all ranks $l = 1, \dots, m$ we have

$$\sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} x_{i,j} \geq \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} y_{i,j} \quad (3)$$

We write $X \geq_{RD} Y$. If inequality (3) is strict for some l , we write $X >_{RD} Y$ for *strict rank dominance*. X is *rank efficient* if there exists no allocation Y such that $Y >_{RD} X$. The intuition is that no agents could trade probability shares in such a way that the overall rank distribution is strictly improved, i.e., the expected number of agents who get a higher ranking choice increases for some rank, while it does not decrease for other ranks. This fairly new efficiency concept is discussed extensively in [Featherstone \(2011\)](#).

4. Hybrid Mechanisms

We now introduce *hybrid mechanisms*, a new concept put forward in this paper. Hybrid mechanisms are convex combinations of other mechanisms, called *components*. The idea is that one component (with better incentive properties) introduces a punishment for misreports, while the other (more efficient) component increases overall efficiency.

4.1. Construction of Hybrid Assignment Mechanisms

For two mechanisms f and g and some *mixing factor* $\beta \in [0, 1]$ we define the β -*hybrid mechanism* as the convex combination $h_\beta = h_\beta(f, g) = (1 - \beta)f + \beta g$, where the allocation $h_\beta(\mathbf{t})$ is the convex combination of the allocation matrices $f(\mathbf{t})$ and $g(\mathbf{t})$. To guarantee that h_β is indeed a well-defined mechanism, we must ensure that $h_\beta(\mathbf{t})$ is an allocation. This is the following Proposition. The proof is in Appendix A.1.

Proposition 1. *The convex combination of allocations is an allocation.*

If f is strategyproof, then h_β is obviously strategyproof for $\beta = 0$. However, for $\beta > 0$, and without any restrictions on g , it is possible that h_β is manipulable. Going forward we primarily analyze for which utility functions the hybrid mechanism h_β is partially strategyproof.

Some properties of hybrid mechanisms are straightforward. First, if h is strategyproof for two utilities of the same type, it is also strategyproof for any convex combination of these utilities: for any $u_i, u'_i \in t_i, t_i \neq t'_i$, $\langle u_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \geq 0$ and $\langle u'_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \geq 0$. Then for a convex combination $u''_i = (1 - \alpha)u_i + \alpha u'_i$ we get

$$\begin{aligned} \langle u''_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle &= (1 - \alpha) \langle u_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \\ &\quad + \alpha \langle u'_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \\ &\geq 0. \end{aligned} \tag{4}$$

Second, if the components f and g are u -strategyproof, then so is h_β . This follows from a similar argument. We have just seen that hybrid mechanisms are indeed well-defined. But what about incentives? If both components are strategyproof, then one could simply use the more efficient of the two components. Unfortunately, this does not help us to achieve more attractive *hybrid* mechanisms. Bogomolnaia and Moulin (2001) have shown that no mechanism can be ordinally efficient, strategyproof, and satisfy equal treatment of equals. Furthermore, rank efficiency and strategyproofness are incompatible. To see this, consider Example 5 in Appendix F, borrowed from Theorem 3 in Featherstone (2011). Thus, we must find a suitable trade-off between strategyproofness and efficiency.

4.2. Uniformly Relatively Bounded Utility (URB)

We will now relax strategyproofness to partial strategyproofness to achieve efficiency improvements. It will later turn out that this allows us to improve the efficiency of mechanisms in the sense of imperfect dominance (see Definition 10 in Section 6.1.3). In contrast to prior work on *approximate strategyproofness*, we will not bound the incentives to misreport for all possible agents. Instead our goal is to design hybrid mechanisms that are partially strategyproof, i.e., strategyproof for a quantitatively constrained subset of the utility space in a given setting.

To constrain the space of utility functions we impose *relative bounds* on the agents' utility functions. We require that an agent's utility for a good is at least some factor $r > 1$ higher than its value for the next less preferred good in its ranking. Additionally, we require that the agent's value for its most preferred good is at most B times the value of its least preferred good. Before we define this formally, we first define what it means for two goods to be adjacent.

Definition 4 (Adjacency). A good y is called a successor of good x in $>$ if

1. $x > y$, and
2. for all goods $j \notin \{x, y\}$ we have $j > x \Leftrightarrow j > y$.

If y is the successor of x , then x is called predecessor of y . We say that x and y are adjacent in the ranking $>$ if one is a successor of the other.

We now define uniform relative boundedness from below and from above separately, and we then combine the two definitions.

Definition 5 (Uniform Relative Boundedness). Let u be a utility in type t with preference order $>$, and let $B \geq r^{m-1}$, $r > 1$.

- u satisfies uniformly relatively bounded indifference with respect to r (denoted $\text{URBI}(r)$) if
 1. for all goods $j \in M$ with successor j' we have $u(j) \geq ru(j')$, and
 2. for the least preferred good j'' under $>$ we have $u(j'') = 1$.
- u is uniformly relatively upper bounded by B (denoted $\text{URUB}(B)$) if

$$\max_{j \in M} u(j) \leq B \min_{j \in M} u(j).$$

- u is uniformly relatively bounded with respect to bounds (r, B) (denoted $\text{URB}(r, B)$) if u satisfies $\text{URBI}(r)$ and $\text{URUB}(B)$.

For ease of notation we will use $\text{URBI}(r, m)$, $\text{URUB}(B, m)$, and $\text{URB}(r, B, m)$ to denote the sets of all utilities over m goods and satisfy the respective uniform relative boundedness property. Figure 2 illustrates the URB-concept: the black line represents one example utility function from $\text{URB}(1.2, 4.0, 4)$; the red lines are upper and lower bounds, each depending on the utility value at the next lower choice.

Note that we require the agents to have utility 1 for their least preferred good. This is of technical importance, but not a real restriction: we analyze mechanisms' incentive properties for utilities satisfying $\text{URBI}(r)$, $\text{URUB}(B)$, $\text{URB}(r, B)$. Thus, we evaluate expressions of the form $\langle u_i, f(t_i, t_{-i})_i - f(t_i, t_{-i})_i \rangle \geq 0$. Nothing important changes in this inequality if we multiply u by some non-negative scalar α or add or subtract a multiple of the vector $\mathbf{1} = (1, \dots, 1)$. This more general concept is captured by the following definition.

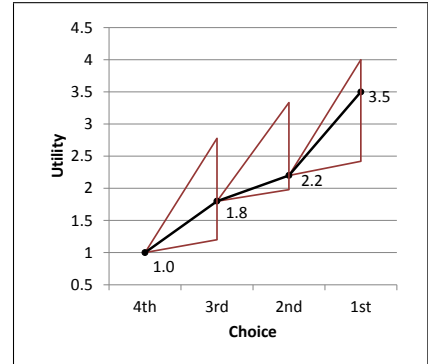


Figure 2: A sample utility function from $\text{URB}(1.2, 4.0, 4)$.

Definition 6 (Quasi-Uniform Relative Boundedness). u' satisfies quasi-URBI(r), quasi-URUB(B), or quasi-URB(r, B) if there exist $\alpha \geq 0$ and $c \in \mathbb{R}$ such that $u' = \alpha u + c\mathbf{1}$ for some u satisfying URBI(r), URUB(B), or URB(r, B), respectively.

All results we prove in this paper for the u -strategyproofness also extend to any $u' = \alpha u + c\mathbf{1}$, where $\alpha \geq 0, c \in \mathbb{R}$. For simplicity we will only state our results for URB.³

Remark 1. If a mechanism is strategyproof on some set URB(r, B, m) in some setting, then no agent with a utility in URB(r, B, m) has an incentive to misreport its preferences. If some other agents' utility does not satisfy uniform relative boundedness with respect to bounds (r, B), this other agent may manipulate. Nonetheless, for all agents with utilities satisfying the condition, it is a dominant strategy to report truthfully.

Remark 2. In the analysis of incentives of hybrid mechanisms, strategyproofness depends on the ability of one component to punish misreports by strictly decreasing expected utility whenever the other component is manipulable. On the other hand, it is not necessary to punish misreports that do not benefit the agent. Therefore, we can relax the URB-requirement: agents can be allowed to be indifferent between goods they rank below their outside option. More generally, in a setting with m goods, if both mechanisms guarantee that the agent will receive one of its m' most preferred goods, with $m' < m$, for all possible joint reports, then it is acceptable for the agent to be indifferent between all goods in its preference ranking at positions $m' + 1$ and below. Thus, we only require agents not to be indifferent between goods they can actually receive via the mechanism (and would also keep).

4.3. Weakly Less Varying Mechanisms

To achieve interesting strategyproofness properties for our hybrid mechanisms, we also need the two component mechanisms to have a particular relationship. For this, we define an invariance monotonicity relation between two mechanisms f and g : if some change of report by agent i changes the allocation for i under g , then the same change of report must also change the allocation for i under f .

Definition 7 (Weakly Less Varying Mechanisms). g is weakly less varying than f , denoted by $g \ll f$, if for any i , any two types $t_i, t'_i \in T$, and any $t_{-i} \in T^{n-1}$ we have

$$g(t_i, t_{-i})_i \neq g(t'_i, t_{-i})_i \Rightarrow f(t_i, t_{-i})_i \neq f(t'_i, t_{-i})_i. \quad (5)$$

g is weakly less varying than f on swaps denoted by $g \ll_{\text{swap}} f$, if (5) holds for any i , any type $t_i \in T$, and any $t'_i \in N_{t_i}$ and any $t_{-i} \in T^{n-1}$.

Recall that the neighborhood of a type t_i is the set of all types with the same preference order, except for a swap of two adjacent goods. Obviously, \ll implies \ll_{swap} , while the contrary may not hold in general.

³A question where quasi-URB may be of importance is in the study of tightness of this concept, but tightness is not covered in this paper.

5. Existence and Computability of Non-trivial Hybrid Mechanisms

We are now ready to prove that non-trivial hybrid mechanisms exist and can be computed.

5.1. Existence Result

Our first main result shows how two mechanisms can be combined by choosing $\beta > 0$ depending on the setting, such that the hybrid is strategyproof on $\text{URB}(r, B, m)$ in the setting.

Theorem 1. *Fix a setting (N, M, \mathbf{q}) . The following statements are equivalent.*

1. *f is a strategyproof mechanism in the setting (N, M, \mathbf{q}) .*
2. *For all $B \geq r^{m-1}, r > 1$ and any mechanism $g \ll_{\text{swap}} f$ there exists $\beta > 0$, such that $h_\beta(f, g)$ is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .*

Proof Idea. We describe the idea for $1. \Rightarrow 2.$ See Appendix B.1 for a formal proof.

Consider a fixed t_{-i} . If a misreport by i changes the outcome of a strategyproof mechanism for i , then it loses expected utility by misreporting. We derive a non-trivial lower bound for this loss. Then we derive an upper bound for the gain i can expect from manipulating any mechanism. In both estimates we use the condition $\text{URB}(r, B)$ to make the bounds uniformly independent of the actual utilities. We then use these bounds to select a non-trivial β such that $\langle u_i, h_\beta(t_i, t_{-i})_i - h_\beta(t'_i, t_{-i})_i \rangle \geq 0$ holds for any $t_i, t'_i \in T, u_i \in t_i \cap \text{URB}(r, B, m)$. This last step is possible due to the assumption $g \ll_{\text{swap}} f$, which implies that whenever an agent manipulates g , the mechanism f changes the allocation and thus reduces its expected utility. Finally, we select the lowest β across all possible reports t_{-i} from other agents. \square

The proof of Theorem 1 uses the following Lemma which is of independent interest. The Lemma shows that when an agent changes its type report by swapping two adjacent goods, then any strategyproof mechanism will only touch the allocation this agent receives for these two goods. No other probabilities may be changed. The proof of Lemma 1 is in Appendix B.1.

Lemma 1. *For any $t_{-i} \in T^{n-1}$, let $t_i \in T, t'_i \in N_{t_i}$, and let $x, y \in M$ be the goods that change position, such that $x >_i y$ and $y >'_i x$. Then for any strategyproof mechanism f we get*

$$(f(t'_i, t_{-i})_i - f(t_i, t_{-i})_i)(x) = (f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i)(y),$$

and for all goods $z \neq x, y$ we have

$$(f(t'_i, t_{-i})_i - f(t_i, t_{-i})_i)(z) = 0.$$

As an immediate consequence of Theorem 1, we obtain the following corollary regarding the existence of hybrid mechanisms that are partially strategyproof.

Corollary 1. *Fix a setting (N, M, \mathbf{q}) . For any strategyproof mechanism f , any $B \geq r^{m-1}, r > 1$, and any mechanism $g \ll_{\text{swap}} f$ there exists $\beta > 0$ such that h_β is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .*

In other words: suppose, a mechanism designer faces an instance of the assignment problem and knows that all agents' utilities satisfy $\text{URB}(r, B)$ for some bounds (r, B) . Suppose further that there is a choice between a strategyproof mechanism f and another mechanism g , where the outcomes of g are more desirable and the hybrid h_β inherits this property. Then in the light of Corollary 1, the designer may do better by using a hybrid mechanism without having to worry about misreports. All she has to do is find a suitable coefficient β for the specific setting. Furthermore, even if the designer cannot be sure about agents' utilities, any agent with a utility in $\text{URB}(r, B, m)$ will have no incentive to deviate from truthful reporting (regardless of the other agents' reports).

Remark 3. *Note that the choice of the mixing factor β depends on the setting (N, M, \mathbf{q}) , the mechanisms f and g , and the bounds (r, B) . However, it is independent of the agents' actual type profile \mathbf{t} , their reports, or their actual utility functions. For example, if β was chosen for setting (N, M, \mathbf{q}) , and now an additional good or a new agent are introduced, then β may no longer guarantee partial strategyproofness of hybrid h_β in the new setting. On the other hand, for a fixed setting (M, N, \mathbf{q}) , mechanisms f, g , and bounds (r, B) , if an agent changes its mind about its type or its utility for some goods, the hybrid remains strategyproof for this agent, as long as the new utility function satisfies $\text{URB}(r, B)$.*

5.2. Computability of the Maximal Mixing Factor

We now consider the computational problem of finding a suitable β such that h_β is strategyproof on $\text{URB}(r, B, m)$ (in a setting). Intuitively, we would like β to be as large as possible, such that any kind of gain from using g is maximized. This value (if it exists) depends on the setting (N, M, \mathbf{q}) , components f and g , and bounds (r, B) . Going forward, we assume some fixed instantiation of these. We define β_{\max} as the largest β such that all incentive constraints are satisfied for all utilities in $\text{URB}(r, B, m)$. This means finding β_{\max} as

$$\begin{aligned} \beta_{\max} &= \max_{\beta} \beta \\ \text{s.t. } &\langle u_i, h_\beta(t_i, t_{-i})_i - h_\beta(t'_i, t_{-i})_i \rangle \geq 0 \\ &\text{for all } i \in N, t_i, t'_i \in T, t_{-i} \in T^{n-1}, \text{ and } u_i \in t_i \cap \text{URB}(r, B, m). \end{aligned} \quad (6)$$

Note that we cannot directly solve the above maximization problem, because using this formalization, the specification has uncountable many constraints, as the set $t_i \cap \text{URB}(r, B, m)$ contains uncountably many utility functions for which inequality (6) must hold. However, we now show that the set of suitable values for β is computable: Proposition 2 yields a finite set of constraints that is sufficient for strategyproofness on $\text{URB}(r, B, m)$, such that an algorithm can then check all constraints and find β_{\max} . The main idea is to exploit the convex structure of $t_i \cap \text{URB}(r, B, m)$ to find a finite subset such that all other utilities are convex combinations of elements of this subset. Intuitively, this subset is the set of vectors that are corners of the convex set $t_i \cap \text{URB}(r, B, m)$. The proof of the following proposition is in Appendix B.2.

Proposition 2. *Fix bounds (r, B) and a setting (N, M, \mathbf{q}) . For type $t \in T$ suppose that*

$$>: j_1 > \dots > j_m$$

ALGORITHM 1: Find β_{\max}

Input: setting (N, M, \mathbf{q}) , mechanisms f, g (with f SP, $g \ll_{\text{swap}} f$), bounds (r, B)

Variables: parameters β_{\max} , agent i , types t_i, t'_i , utility u_i , type collection t_{-i}

begin

```
   $\beta_{\max} \leftarrow 1$ 
  for  $i \in N, t_{-i} \in T^{n-1}$  do
     $f \leftarrow f(\cdot, t_{-i})_i; g \leftarrow f(\cdot, t_{-i})_i$ 
    for  $t_i, t'_i \in T, u_i \in \eta(t_i, r, B, m)$  do
      if  $\langle u_i, f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i \rangle < \langle u_i, g(t'_i, t_{-i})_i - g(t_i, t_{-i})_i \rangle$  then
         $\beta_{\max} \leftarrow \min \left( \beta_{\max}, \frac{\langle u_i, f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i \rangle}{\langle u_i, f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i \rangle - \langle u_i, g(t_i, t_{-i})_i - g(t'_i, t_{-i})_i \rangle} \right)$ 
      end
    end
  end
end
```

is the corresponding preference order. Then let

$$\eta(t, r, B, m) = \left\{ u \in t : \exists k \in \{1, \dots, m\} \text{ with } u(j_l) = \begin{cases} r^{m-l}, & \text{if } l \geq m - k + 1, \\ Br^{1-l}, & \text{if } l \leq m - k \end{cases} \right\}^4.$$

Then for any mechanism h the following statements are equivalent.

1. h is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .
2. For any agent i , any $t_i, t'_i \in T$, any $t_{-i} \in T^{n-1}$, and any $u_i \in \eta(t_i, r, B, m)$ we have

$$\langle u_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \geq 0. \quad (7)$$

Proof Idea. We show that for any type t_i , $\text{convex}(\eta(t_i, r, B, m)) = t_i \cap \text{URB}(r, B, m)$, where $\text{convex}(A)$ denote the convex hull of A . Showing convexity of $t_i \cap \text{URB}(r, B, m)$ is straightforward. The inclusion $\eta(t_i, r, B, m) \subseteq t_i \cap \text{URB}(r, B, m)$ is trivial, and hence, $\text{convex}(\eta(t_i, r, B, m)) \subseteq t_i \cap \text{URB}(r, B, m)$. To show “ \supseteq ” we prove by induction over the number of goods m that any element of $t_i \cap \text{URB}(r, B, m)$ can be represented as a convex combination of elements of $\eta(t_i, r, B, m)$. \square

Proposition 2 has important consequences: it allows us to formulate Algorithm 1 to compute β_{\max} , i.e., the maximum value of β such that the hybrid mechanism is strategyproof on $\text{URB}(r, B, m)$. The following Proposition ensures that Algorithm 1 is indeed computable: it terminates for all valid inputs (*completeness*), and if it terminates, the result is correct (*correctness*). The proof is in Appendix B.2.

⁴This special subset is the set of utilities in t , where the utility for the last choice is 1, and the value for consecutive elements increases by a factor of r , except for possible one pair. At this pair the factor increase can be $\frac{B}{r^{m-2}}$ instead.

Proposition 3. *If f and g are computable, Algorithm 1 is correct and complete, i.e., the algorithm terminates on all valid inputs, and for a setting (N, M, \mathbf{q}) , mechanisms f, g with f SP, $g \ll_{\text{swap}} f$, and bounds (r, B) , Algorithm 1 finds a β_{\max} such that $h_{\beta_{\max}}$ is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .*

A consequence of Proposition 3 is that the set of suitable values for β has a very simple structure: it is an interval $[0, \beta_{\max}]$. When decreasing β below β_{\max} , the mechanism h_{β} remains strategyproof on $\text{URB}(r, B, m)$. On the other hand, increasing β above β_{\max} always introduces manipulability of the hybrid mechanism in the sense that an agent with a certain utility function in $\text{URB}(r, B, m)$ get an incentive to misreport. β_{\max} is the value computed by Algorithm 1. This means that the algorithm indeed finds the highest possible β such that the hybrid is strategyproof on $\text{URB}(r, B, m)$.

Corollary 2. *Given setting (N, M, \mathbf{q}) , mechanisms f, g with f SP, $g \ll_{\text{swap}} f$, and bounds (r, B) , Algorithm 1 computes a $\beta_{\max} \in (0, 1]$ such that*

1. h_{β} is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) for all $\beta \leq \beta_{\max}$, and
2. h_{β} is not strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) for all $\beta_{\max} < \beta \leq 1$.

Proof. Algorithm 1 computes a β_{\max} where all binding constraints are upper bounds for β_{\max} , hence a lower β would not violate any constraint. On the other hand a higher β will violate at least one of the constraints, namely the constraint that was ultimately binding. Thus, there exist $t_i, t'_i \in T, t_{-i} \in T^{n-1}, u_i \in \text{URB}(r, B, m)$ such that $\langle u_i, h_{\beta}(t_i, t_{-i})_i - h_{\beta}(t'_i, t_{-i})_i \rangle < 0$, i.e., h_{β} is not strategyproof on $\text{URB}(r, B, m)$. \square

6. Efficiency of Hybrid Mechanisms

We have previously expressed hope that hybrid mechanisms can improve efficiency over their less efficient component. We now formalize this intuition. Recall that a mechanism satisfies a certain efficiency concept if all outcomes are guaranteed to satisfy the efficiency concept with respect to the reported preferences. For the clear-cut case when the outcome of g always dominates the outcome of f in some sense, the answer to this question is straightforward and positive. The proof for the next Proposition can be found in Appendix C.

Proposition 4. *For $X, Y \in \mathcal{X}$, X ex-post efficient, let $Z = (1 - \beta)X + \beta Y$ with $\beta \in (0, 1]$. Then*

1. Y ex-post efficient $\Rightarrow Z$ ex-post efficient,
2. Y (strictly) ordinally dominates $X \Rightarrow Z$ (strictly) ordinally dominates X ,
3. Y (strictly) rank dominates $X \Rightarrow Z$ (strictly) rank dominates X .

Proposition 4 yields two insights. First, if both mechanisms are ex-post efficient, then so is the hybrid. This is good, because one can guarantee ex-post efficiency in conjunction with strategyproofness by using *RSD*. Second, whenever the outcome of g dominates the outcome of f in some sense, then the outcome of the hybrid dominates the outcome of f in the same sense. But what can we say about cases when this dominance relationship is not satisfied for all type profiles? This requires the extension of the dominance-relation from outcomes to mechanisms.

6.1. Efficiency Comparison of Mechanisms by Dominance

In this section, we introduce and discuss three notions for comparing mechanisms by their efficiency. For all three notions we consider ordinal dominance and rank dominance. To keep it short, we will drop the identifier of the specific dominance concept (ordinal or rank) if a statement holds for any of the two.

6.1.1. Ordinal (or Rank) Dominance

A canonical notion of a *more efficient* mechanism arises if we compare them outcome by outcome.

Definition 8 (Mechanism Dominance). h ordinally (or rank) dominates f if for all type profiles \mathbf{t} , $h(\mathbf{t})$ ordinally (or rank) dominates $f(\mathbf{t})$.

h strictly ordinally (or rank) dominates f if h ordinally (or rank) dominates f , and for some type profile $\tilde{\mathbf{t}}$, $h(\tilde{\mathbf{t}})$ strictly ordinally (or rank) dominates $f(\tilde{\mathbf{t}})$.

We adopt the notation

$$h \geq_{OD} f, h >_{OD} f, h \geq_{RD} f, h >_{RD} f$$

to denote the respective dominance for mechanisms.

For ordinal dominance this is the same notion of dominance as used for example in [Erdil \(2011\)](#). It is strong in a sense that if h strictly dominates f , all agents would unambiguously agree that h yields at least weakly better outcome (under truthful reports). This agreement is independent of their specific utility functions. For rank dominance it is strong in the sense that the rank distribution of h is socially preferable to that of f if all agents are weighted equally. On the other hand, the comparison may not be practical since it allows comparison for only a small set of (pairs of) mechanisms. If the outcome of h dominates the outcome of f , except at a single type profile, neither mechanism can be viewed as more efficient from the perspective of ordinal or rank dominance.

The following example shows that this is indeed a problem: non of the two concepts ordinal or rank efficiency guarantee that the allocation dominates any less efficient allocation.

Example 2. Existence of inefficient, but undominated allocations: Consider a setting with 4 goods $\{a, b, c, d\}$, each with unit capacity, and 4 agents $\{1, 2, 3, 4\}$. Let the agents'

preferences be

$$\begin{aligned} >_i, i = 1, 2 & : a > b > c > d, \\ >_3 & : b > c > a > d, \\ >_4 & : a > d > c > b. \end{aligned}$$

Then let

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Y = \frac{1}{8} \begin{pmatrix} 3 & 0 & 0 & 5 \\ 0 & 1 & 7 & 0 \\ 1 & 7 & 0 & 0 \\ 4 & 0 & 1 & 3 \end{pmatrix}.$$

X is rank efficient: one first choice (good a to agent 1) and 3 second choices are allocated, i.e., the rank distribution is $(1, 4, 4, 4)$. Any rank efficient allocation must give good d to agent 4. Hence, to increase the number of allocated first choices, agent 3 must get b . But then either agent 1 or agent 2 get c , their third choice. This allocation cannot rank dominate X , because under X no third choice is allocated. Rank efficiency of X implies ordinal and ex-post efficiency.

Y is not even ex-post efficient: agent 3 cannot receive good c in any ex-post efficient allocation, since it could always trade with the agent who received d . This implies that Y is neither ordinally nor rank efficient.

However, Y is not ordinally or rank dominated by X : the rank distribution of Y is $\frac{1}{8}(14, 18, 27, 32)$, where in particular more than 1 first choice are allocated (in expectation). Y allocates a non-trivial share of a to agent 4, which is agent 4's first choice, but X does not. Hence, Y is not ordinally dominated by X either.

With Example 2 in mind, it is clear that not all mechanisms are comparable with respect to dominance as defined in Definition 8.

Proposition 5. *There exist mechanisms which are not comparable with respect to dominance.*

Proof. We can construct mechanisms h and f such that the outcomes of h always dominate the outcomes of f , except for preferences as in Example 2. For these preferences, h chooses X , while f chooses Y . Then neither of the mechanisms dominates the other weakly or strictly in the sense of Definition 8. \square

6.1.2. Concept Dominance

We learn from Proposition 5 that a canonical extension of the dominance relation to mechanisms may be too restrictive. We now turn to a weaker notion of dominance, motivated by the intuition that undominated allocations are in some sense more appealing than dominated allocation.

The canonical dominance notion requires outcomes of h to *at least weakly and sometimes strictly dominate* those of f . We weaken the requirement in the sense that outcomes of h must be *at least as efficient and sometimes more efficient* than those of f , but we do not

require pairwise dominance of the individual outcomes. This notion is consistent with the hierarchy of efficiency concepts ex-post, ordinal, and rank efficiency: it captures the intuition that an ordinally efficient or rank efficient outcome is somehow more appealing, because the mechanism performs as well as possible by finding undominated allocations as opposed to potentially dominated ones.

Definition 9 (Concept Dominance). h ordinally (or rank) concept dominates f if for every type profile \mathbf{t} we have

$$f(\mathbf{t}) \text{ ordinally (or rank) efficient} \Rightarrow h(\mathbf{t}) \text{ ordinally (or rank) efficient.}$$

Concept dominance is strict if h concept dominates f and for some type profile $\tilde{\mathbf{t}}$, $h(\tilde{\mathbf{t}})$ is ordinally (or rank) efficient, but $f(\tilde{\mathbf{t}})$ is not.

We introduce the notation

$$h \geq_{COD} f, h >_{COD} f, h \geq_{CRD} f, h >_{CRD} f.$$

Many more pairs of mechanisms are comparable by concept dominance, e.g., the mechanisms constructed in the proof of Proposition 5 are clearly comparable. Also, if h ordinally (or rank) dominates f weakly, then the respective concept dominance also holds weakly in the same direction.

Unfortunately, this is already the end of the good news. First, while h may strictly ordinally (or rank) dominate f , this strictness does not always translate into strict concept dominance. Second, h may strictly concept dominate f , even though all outcomes of f dominate the outcomes of h (even strictly for some), except at a single type profile. What is more, at this type profile, agents may have utilities such that all of them unambiguously agree that the allocation under f is better than under h . The following examples illustrate these drawbacks.

Example 3. Existence of strictly dominant, but concept-equivalent allocations: Consider the setting from Example 2 with 4 agents and 4 goods with unit capacity. Suppose that the agents' preferences are

$$\begin{aligned} >_i, i = 1, 2 & : a > b > c > d, \\ >_i, i = 3, 4 & : a > c > b > d. \end{aligned}$$

and consider the allocations

$$Y_1 = \frac{1}{8} \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}, Y_2 = \frac{1}{8} \begin{pmatrix} 2 & 3 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 \end{pmatrix}, X_1 = \frac{1}{8} \begin{pmatrix} 2 & 4 & 0 & 2 \\ 2 & 4 & 0 & 2 \\ 2 & 0 & 4 & 2 \\ 2 & 0 & 4 & 2 \end{pmatrix}.$$

Note that $X_1 >_{OD} Y_1 >_{OD} Y_2$ and $X_1 >_{RD} Y_1 >_{RD} Y_2$. Furthermore, X_1, Y_1, Y_2 are ex-post efficient: each is a convex combination of ex-post efficient deterministic allocations (this shown in Remark 5 in Appendix F). Y_1 and Y_2 are not ordinally efficient and hence neither rank efficient, since both are dominated by X_1 . Thus, we found a merely ex-post efficient allocation that ordinally and rank dominates another ex-post efficient allocation.

The next Proposition shows the *undecidedness* of concept dominance.

Proposition 6. *There exist mechanisms h, f such that h strictly dominates f in the sense of Definition 8, but h does not strictly concept dominate f , and f and h weakly concept dominate each other.*

Proof. Suppose, mechanisms h and f always produce the same ex-post efficient allocations, except for types \mathbf{t} when $f(\mathbf{t})$ is ex-post efficient, but not ordinally efficient. Then the allocation from $h(\mathbf{t})$ is chosen to ordinally and rank dominate $f(\mathbf{t})$, but also as ordinally inefficient. Example 3 shows that this is possible. Then h and f are equivalent with respect to concept dominance, because whenever $f(\mathbf{t})$ is ordinally or rank efficient, then so is $h(\mathbf{t})$, but the allocations never satisfy different efficiency concepts. This implies $h \geq_{COD} f$, $f \geq_{COD} h$, $h \geq_{CRD} f$, and $f \geq_{CRD} h$.

On the other hand, for at least one type profile $\tilde{\mathbf{t}}$, $h(\tilde{\mathbf{t}})$ strictly ordinally and rank dominates $f(\tilde{\mathbf{t}})$, and for all other profiles at least $h(\mathbf{t}) \geq_{OD} f(\mathbf{t})$ and $h(\mathbf{t}) \geq_{RD} f(\mathbf{t})$. This implies $h >_{OD} f$ and $h >_{RD} f$ with respect to the strong notion of efficiency comparison from Definition 8. \square

This shows the failure of concept dominance to detect a clear dominance of h over f . The next example can be used to illustrate that concept dominance may actually make fatal errors, i.e., it can give strict preference to a mechanism that may be less attractive in a certain sense.

Example 4. Existence of inefficient allocations that all agents prefer: *We consider the setting and preferences from Example 3, and the allocation*

$$X_2 = \frac{1}{6} \begin{pmatrix} 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

X_2 is ex-post efficient (see Remark 5 in Appendix F), but not ordinally or rank efficient: X_2 is rank dominated by X_1 , and agents 1 and 3 would agree to trade shares of b and c . Thus, X_1 concept dominates X_2 .

Suppose now that agents 1 and 2 have utility $u = (400, 12, 8, 4)$, and agents 3 and 4 have utility $u' = (400, 392, 396, 4)$ for goods a, b, c, d , respectively. Then the difference in expected utility for agents $i = 1, 2$ is

$$\langle u, (X_1 - X_2)_i \rangle = 107 - 138 < 0,$$

and for agents $i = 3, 4$ it is

$$\langle u', (X_1 - X_2)_i \rangle = 299 - 330 < 0.$$

Thus, all agents prefer allocation X_2 to X_1 . Note that this results is stronger than simply stating that X_1 does not dominate X_2 : ordinal non-dominance only implies that a utility profile exists such that at least one agent prefers X_2 , but in the example all agents prefer X_2 .

The following proposition formalizes the way in which concept dominance can make arguable wrong decisions.

Proposition 7. *There exist mechanisms h, f , such that f strictly concept dominates h , but*

- *For some type profile $\tilde{\mathbf{t}}$, there exists a utility profile \mathbf{u} such that all agents strictly prefer the allocation $h(\tilde{\mathbf{t}})$ to the allocation $f(\tilde{\mathbf{t}})$.*
- *For all type profiles $\mathbf{t} \neq \tilde{\mathbf{t}}$, the allocation $h(\mathbf{t})$ dominates the allocation $f(\mathbf{t})$, and the dominance is strict for some type profiles.*

Proof. Consider mechanisms h and f as constructed in the proof of Proposition 5, where $h(\mathbf{t})$ dominates $f(\mathbf{t})$ whenever possible, but never satisfies a higher efficiency concept. Change this mechanism for some type profile $\tilde{\mathbf{t}}$ such that $h(\tilde{\mathbf{t}})$ is not ordinally or rank efficient, but $f(\tilde{\mathbf{t}})$ is. Example 4 shows that there may exist utilities such that all agents prefer the outcome of h under $\tilde{\mathbf{t}}$. Nevertheless, $f \succ_{COD} h$ or $f \succ_{CRD} h$. \square

So even though h is without doubt the better choice in almost all cases and may even be the better choice for the exception, concept dominance will strictly recommend f .

6.1.3. Imperfect Dominance

We now consider a third alternative for comparison of mechanisms' efficiency. Again we start with some notion of dominance, but alter the intuition: h should be preferred to f if, whenever the outcomes are comparable by the dominance-notion, then the outcome of h is better. Essentially, this is equivalent to the strong notion of dominance from Section 6.1.2, but deliberately ignoring cases where no decision can be made.

Definition 10 (Imperfect Dominance). *h ordinally (or rank) imperfectly dominates another mechanism f if for all type profiles \mathbf{t} , $f(\mathbf{t})$ does not strictly ordinally (or rank) dominate $h(\mathbf{t})$.*

h strictly ordinally (or rank) imperfectly dominates f if h ordinally (or rank) concept dominates f , and there exists a type profile $\tilde{\mathbf{t}}$ such that $h(\tilde{\mathbf{t}})$ strictly ordinally (or rank) dominates $f(\tilde{\mathbf{t}})$.

The respective notation is

$$h \geq_{IOD} f, h \succ_{IOD} f, h \geq_{IRD} f, h \succ_{IRD} f.$$

Imperfect dominance does not suffer the aforementioned problems of concept dominance. On the mechanisms from Proposition 6, imperfect dominance decides in favor of h , where concept dominance is undecided. On the mechanisms presented in Proposition 7, imperfect dominance actually decides strictly in favor of h , where concept dominance picks the arguably less desirable f .

There exist mechanisms that are incomparable with respect to imperfect dominance, e.g., if the outcomes of h sometimes strictly dominate f , and sometimes the reverse is true. Concept dominance may make a decision here (even strictly), however, it depends on the utility functions of the agents whether this decisiveness is actually attractive.

Another limitation of imperfect dominance is that the relation is not transitive, i.e., $g \succeq_{IOD} h$ and $h \succeq_{IOD} f$ does not imply $g \succeq_{IOD} f$. This is because the outcomes $h(\mathbf{t})$ may be incomparable to $g(\mathbf{t})$ or $f(\mathbf{t})$, but $f(\mathbf{t})$ may dominate $g(\mathbf{t})$. Therefore, \succeq_{IOD} does not induce a partial ordering on the space of mechanisms. The imperfect rank dominance relation has the same problem.

Hence with all due care, we employ imperfect dominance to compare the efficiency of hybrid mechanisms and their components.

6.2. Efficiency Comparison of Hybrid Mechanisms

We find that if one considers an ex-post efficient mechanism f as the first component of a hybrid mechanism h , and chooses a g that imperfectly dominates f , then h also imperfectly dominates f . This is Proposition 8, which is proven in Appendix C.1.

Proposition 8. *For mechanisms f, g , let $h_\beta = (1 - \beta)f + \beta g$ be a hybrid with $\beta \in (0, 1]$. Then*

1. $g \succeq_{IOD} f \Rightarrow h \succeq_{IOD} f$ and $g \succeq_{IRD} f \Rightarrow h \succeq_{IRD} f$,
2. $g \succ_{IOD} f \Rightarrow h \succ_{IOD} f$ and $g \succ_{IRD} f \Rightarrow h \succ_{IRD} f$.

This guarantees that when g weakly imperfectly dominates f , then so does h , and if the imperfect dominance of g is strict, then so is the imperfect dominance of h . We also learn that the outcome of f never strictly dominates the outcome h . Note however that we cannot guarantee concept dominance of the hybrid mechanism over f , since the convex combination of ordinally (or rank) efficient allocations may not be ordinally (or rank) efficient. Hence, some allocations produced by h may be ordinally (or rank) inefficient, i.e., they may be dominated by other allocations than those from f .

7. Hierarchies of Efficiency and Manipulability

We have established incentives as well as efficiency improvements for hybrid mechanisms separately. Concerning the trade-off between strategyproofness and efficiency, we find that as the mixing factor β increases, the resulting hybrid mechanisms become more efficient, but also more manipulable. In the following, suppose a setting (N, M, \mathbf{q}) , a strategyproof mechanism f , a mechanism g that imperfectly dominates f , $g \ll_{\text{swap}} f$, and g is manipulable in the setting. Furthermore, let $0 \leq \beta < \beta' \leq 1$.

7.1. Efficiency Hierarchy

The following Proposition shows that $h_{\beta'}(f, g)$ imperfectly dominates $h_\beta(f, g)$, i.e., efficiency increases as the mixing factor increases.

Proposition 9. *Let f be strategyproof and $g \ll_{\text{swap}} f$, and let g (strictly) imperfectly dominates f . Then for $0 \leq \beta < \beta' \leq 1$, h_β (strictly) imperfectly dominates $h_{\beta'}$.*

Proof. Analogous to the proof of Proposition 8, one can show that g imperfectly dominates $h_\beta(f, g)$. Therefore, we can apply Proposition 8 to mechanisms $h_\beta(f, g)$ and g , and with mixing parameter $\frac{\beta' - \beta}{1 - \beta} \in (0, 1]$. This yields imperfect dominance of $h_{\beta'}(f, g)$ over $h_\beta(f, g)$. \square

7.2. Manipulability Hierarchy

While $h_{\beta'}(f, g)$ imperfectly dominates $h_\beta(f, g)$, it is also less robust to manipulation. To compare mechanisms by their degree of vulnerability to manipulation, Pathak and Sönmez (2013) have recently introduced a framework that allows for the comparison of arbitrary mechanisms. Using their framework and our parametrization of hybrid mechanisms via the mixing factor β , we can describe a hierarchy of hybrid mechanisms that are strictly more manipulable for strictly larger β s. We next describe the intuition behind their framework. The formal definitions are repeated in Appendix G.

Informally, a mechanism h' is *as strongly and intensely manipulable as* mechanism h if any agent who can manipulate h can also manipulate h' , and the increase in expected utility from manipulating h' is at least as high as from manipulating h . A mechanism h' is *intensely and strongly more manipulable than* h if it is as intensely and strongly manipulable as h , and there exists a utility such that an agent with this utility could manipulate h' , but not h .

Proposition 10. *Let f be strategyproof and $g \ll_{\text{swap}} f$, and let g be manipulable for some setting (M, N, \mathbf{q}) . Then for $0 \leq \beta < \beta' \leq 1$, $h_{\beta'}$ is intensely and strongly more manipulable than h_β .*

Proof Idea. We first show that the constraints in Algorithm 1 are continuous in the bounds. By the interim value theorem, we find bounds (r, B) such that β is equal to the β_{\max} found by Algorithm 1 applied to (r, B) . Because $\beta < 1$, some constraint in Algorithm 1 was binding, which implies that we can find a utility u in $\text{URB}(r, B, m)$ for which h_β is strategyproof in the setting (N, M, \mathbf{q}) , but $h_{\beta'}$ is not. See Appendix D.1 for a formal proof. \square

8. Hybrid Instantiations: Mixing *RSD* with *PS* and *RV*

In this section, we instantiate the theoretical results from the previous sections with a non-trivial hybrid mechanism that uses Random Serial Dictatorship (*RSD*) and Probabilistic Serial (*PS*) as component mechanisms. The main technical challenge to obtain this result is to show that $PS \ll_{\text{swap}} RSD$. For the Rank Value mechanism (*RV*), we get a negative result. We show that for certain choices of rank valuation, $RV \not\ll_{\text{swap}} RSD$, and that *RSD* and *RV* cannot be combined to non-trivial hybrid mechanisms that are partially strategyproof.

8.1. Mixing *RSD* and *PS*

Random Serial Dictatorship (RSD) mechanisms constitute a whole class of mechanisms that work as follows: first, the mechanism randomly draws an ordering of the agents according to some previously fixed distribution. The first agent from the ordering gets to pick its favorite

good. Next, the second agent can choose amongst the remaining available goods. This continues until all agents have picked a good. The non-deterministic allocation matrix produced by RSD is determined *after* all reports are submitted, but *before* the specific ordering of agents is drawn. The choice of distribution instantiates a specific mechanism.

All RSD mechanisms are ex-post efficient, strategyproof, and nonbossy (nonbossy is shown in Lemma 7 in Appendix E.1). Most commonly, a uniform distribution over orderings is used, which makes the specific mechanism RSD^{unif} anonymous⁵. Algorithm 2 in Appendix H implements RSD^{unif} .

Probabilistic Serial (PS) mechanisms also constitute a whole class of mechanisms that work as follows (Bogomolnaia and Moulin, 2001): Implicitly, the mechanisms treat all goods *as if* they were divisible. Agents begin “consuming” probability shares of their favorite goods. When a good’s capacity is exhausted, the agents from this good move on to their respective next favorite good. This continues until all agents have accumulated a total of 1 in probability shares. Agents are endowed with *eating speed functions*⁶ $s_i(\tau) \geq 0$, and each agent consumes $\int_0^{\tau'} s_i(\tau) d\tau$ shares by time τ' . The probability shares accumulated by an agent constitute its probabilistic allocation.

All PS mechanisms are ordinally efficient and nonbossy (see Theorem 1 in Kesten and Ekici (2012)), but may not be strategyproof. For the analysis, it is important to note that the incentives to manipulate depend on agents’ cardinal utilities.⁷ A popular choice of eating speed functions are uniform eating speeds, i.e., $s_i(\tau) = 1$ for all $i \in N, \tau \in [0, 1]$, and the resulting mechanism PS^{unif} is anonymous (Bogomolnaia and Moulin, 2001).⁸ Algorithm 3 in Appendix H implements PS^{unif} .

To simplify notation, we will omit the superscript “unif” for both mechanisms and refer to RSD^{unif} and PS^{unif} as RSD and PS , respectively, except in formulations of results.

We now show that for any setting (N, M, \mathbf{q}) , PS is weakly less varying on swaps than RSD , i.e., if an agent i swaps two goods in its report, and due to this PS changes i ’s allocation, then the same report change must also change i ’s allocation under RSD . The formal proof is in Appendix E.1.

Theorem 2. *For any setting (N, M, \mathbf{q}) , PS^{unif} is weakly less varying on swaps than RSD^{unif} , i.e., $PS^{\text{unif}} \ll_{\text{swap}} RSD^{\text{unif}}$.*

Proof Idea. Suppose agent 1 of type t considers to misreport as a neighboring type $t' \in N_t$ by swapping two adjacent goods in its preference report, x and y , say. We must show that this

⁵To the best of our knowledge it is an open question whether ex-post efficiency, strategyproofness, and anonymity uniquely characterize RSD^{unif} .

⁶ s_i is non-negative and chosen so that $\int_0^1 s_i(\tau) d\tau = 1$, i.e., consumption ends at time 1.

⁷Note that PS satisfies the much weaker concept of *weak strategyproofness*, i.e., for any number of agents n , goods m , capacities \mathbf{q} , eating speed functions s , and any type $t \in T$ there exists a utility $u \in t$ such that PS is u -strategyproof (see Bogomolnaia and Moulin (2001)).

⁸Hashimoto et al. (2013) characterize PS^{unif} as the unique mechanism that is ordinally efficient and ordinally fair. *Ordinal fairness* is satisfied if for an agent i , who receives non-trivial shares of some good x , the probability of receiving some good y it weakly prefers to x is no larger than any other agent’s probability of receiving a good that this agent weakly prefers to x .

swap either has no influence on its allocation under PS , or this swap changes the allocation under RSD as well. The proof proceeds in three steps:

1. Characterize the report profiles and misreports for which PS changes the allocation: PS changes the allocation if and only if neither x nor y are exhausted when agent 1 begins consuming at x (under truthful report).
2. Characterize the report profiles and misreports for which RSD changes the allocation: RSD changes the allocation if and only if there exists an ordering of the agents such that all agents preceding agent 1 in this ordering would remove all goods that agent 1 prefers to x and y , but neither x nor y .
3. Show that the criterion in 1. implies the criterion in 2.

3. can be shown by recursively selecting agents for the specific ordering and exploiting the fact that x and y are not exhausted under PS when agent 1 gets to them. \square

Theorem 2 allows us to now state our most tangible result in this paper: we can combine RSD (for uniform distributions over orderings) and PS (for uniform eating speeds) to non-trivial hybrid mechanisms that are ex-post efficient, strictly ordinally imperfectly dominate RSD , and are partially strategyproof on $URB(r, B, m)$ in a given setting.

Corollary 3. *Given a setting (N, M, \mathbf{q}) , where $m \geq 4$, and bounds (r, B) , there exists a $\beta > 0$ such that $h_\beta(RSD^{\text{unif}}, PS^{\text{unif}})$ is strategyproof on $URB(r, B, m)$ in the setting and ex-post efficient. Furthermore, $h_\beta(RSD^{\text{unif}}, PS^{\text{unif}}) >_{IOD} RSD^{\text{unif}}$.*

Proof. RSD is strategyproof, and in Theorem 2 we have shown that $PS \ll_{\text{swap}} RSD$. Thus, RSD and PS satisfy the criteria needed to apply Corollary 1, which gives us the first part of the corollary. For the second part, observe that $PS \geq_{IOD} RSD$, since the outcomes of PS are never ordinally dominated by any allocation. Example 1 from Section 3.4 shows that for $m \geq 4$ type profiles exist for which the outcome of PS strictly ordinally dominates the outcome of RSD . This yields $PS >_{IOD} RSD$, and using Proposition 8 we get the desired imperfect dominance. \square

Remark 4. *Kojima and Manea (2010) have shown that for a fixed set of goods, PS^{unif} is strategyproof in settings where the capacities \mathbf{q} of the goods are sufficiently large. Thus, the parameter β_{max} found by Algorithm 1 for mixing PS^{unif} and RSD^{unif} will be 1 if applied to a setting where enough copies of each good are available. This will be the case for any bounds (r, B) . In the light of this result, our hybrids $h_\beta(RSD^{\text{unif}}, PS^{\text{unif}})$ become particularly useful for settings where the capacities are small, and PS^{unif} is not strategyproof.*

8.2. Impossibility Result for Mixing RSD and RV

After having seen an actual instantiation of non-trivial hybrid mechanisms mixing RSD and PS , we now turn to a negative result that demonstrates that not all mechanisms can be

usefully mixed, which is the case for *RSD* and *RV*. Here, *RV* denotes the class of *Rank Value* mechanisms (Featherstone, 2011). The allocation is determined by

$$X = \arg \max_{X \text{ allocation}} \sum_{i \in N, j \in M} x_{i,j} v_{\text{rank}_i(j)}. \quad (8)$$

The choice of a *rank valuation* $v = (v_1, \dots, v_m)$ with $v_k > v_{k+1}$ for all k yields an instantiation of the many possible *RV* mechanisms, denoted $v - RV$. An entry v_k is interpreted as some *social value* for assigning an agent its k th choice, though it does not need to reflect actual cardinal utilities. The $\arg \max$ may contain more than one rank efficient allocation. For a well-defined mechanism, a tiebreaker must be chosen⁹. *RV* mechanisms are not strategyproof, but they are rank efficient, which is a stronger requirement than ordinal efficiency. One might hope that hybrid mechanisms mixing *RSD* and *RV* yields imperfectly rank dominant and partially strategyproof mechanisms.

This hope is not unwarranted in a sense that there might exist rank valuations and tiebreakers for which the *RV* mechanism is weakly less varying than *RSD*. However, for a rather large class of rank valuations, we can prove the opposite. Thus, Corollary 1 cannot be applied to obtain partially strategyproof non-trivial hybrids in general.

Featherstone (2011) uses the rank valuation

$$v = (100, 80, 50, 35, 15, 10, 5, 3, 2, 1, 0.5)$$

for a setting with 11 goods in numerical evaluations. Observe that $v_3 - v_4 = 15 < 20 = v_4 - v_5$. This motivates the definition of the class of rank valuations with non-decreasing increments: v is said to have *non-decreasing increments* if for some $k \in \{1, \dots, m-2\}$ we have $v_k - v_{k+1} < v_{k+1} - v_{k+2}$. For valuations with non-decreasing increments, a non-trivial hybrid of *RV* and *RSD* cannot be partially strategyproof. This is the result of the following proposition. The proof is in Section E.2 of the Appendix.

Proposition 11. *If v is a rank valuation with non-decreasing increments, then for any number of goods $m \geq 3$ there exists a setting (N, M, \mathbf{q}) and preference profile $\succ = (\succ_1, \succ_{-1})$ such that agent 1 can beneficially manipulate *RV*, but *RSD*^{unif} is invariant to this manipulation.*

Proof Idea. For the specific choice of rank valuation we construct a preference profile such that *RV* offers an unambiguous manipulation to agent 1 (regardless of its actual utility). Then we show that for *RSD* does not react to this manipulation in the sense that it does not change the allocation of agent 1 when it changes its report from the truth to the manipulation. \square

Intuitively, this means that the hybrid mechanism is manipulable for an agent of some type (independent of its specific utility within that type): any share of *RSD* in the hybrid cannot compensate the incentives to manipulate the remaining share of *RV*. This does not mean that no other strategyproof mechanism exists that could always be usefully mixed with *RV*, but

⁹One option is to find all deterministic rank efficient allocations in the $\arg \max$, then assign uniform probabilities (see Featherstone (2011)). Example 5 illustrates the manipulability of *RV*, for any choice of rank valuation v and any tiebreaker.

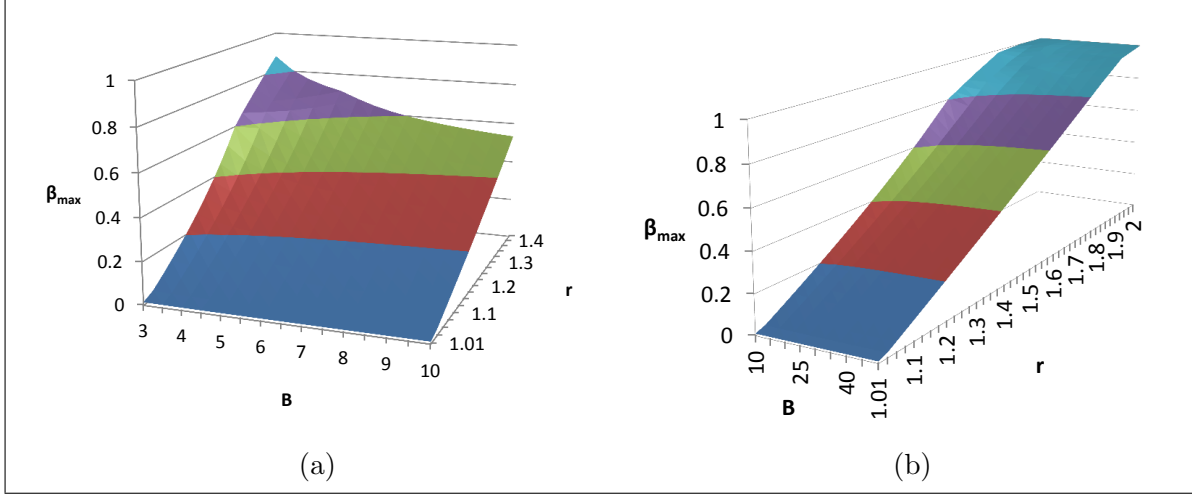


Figure 3: Numerical analysis for $h_{\beta}(RSD, PS)$. The plots show β_{\max} on the vertical axis for a setting with 4 agents and 4 goods with unit capacity: (a) $B \in [3, 10]$ and $r \in [1.01, 1.425]$; (b) $B \in [10, 50]$ and $r \in [1.01, 2]$.

it will not be RSD . This is unfortunate, since RSD is a canonical choice for a strategyproof and ex-post efficient mechanism. It remains an open research question whether for any rank valuations RV is in fact weakly less varying than RSD , and it would also be interesting to explore other mechanisms that can be mixed with RV .

9. Numerical Results and Computational Complexity

In this section, we provide numerical results for hybrids mixing RSD and PS to demonstrate the range of the maximal mixing factor β_{\max} . To this end, we implemented Algorithm 1 and computationally determined β_{\max} in a tractable setting, varying the bounds (r, B) .

9.1. Maximal Mixing Factor β_{\max} for 4 Agents and 4 Goods

Figure 3 shows plots of β_{\max} on the y-axis, for a setting with 4 agents and 4 goods with unit capacity. On the left, we let $B \in [3, 10]$ and $r \in [1.01, 1.4]$. On the right we let $B \in [10, 50]$ and $r \in [1.01, 2]$. We see that the range of β_{\max} is large, and is primarily driven by the bound r . To give just one example, we can see in the left graph that for $B = 10$ and $r = 1.4$, β_{\max} is 50%. This means that if agents value their most preferred good at most 10 times as high as their least preferred good, and each good is worth to them at least 40% more than the next alternative, the hybrid mechanism $h_{0.5}(RSD, PS)$ is strategyproof for these agents (in settings with 4 agents and 4 goods with unit capacity).

The plots in Figure 3 suggest that β_{\max} is linearly dependent on r . We can see that β_{\max} eventually reaches 1 when r is sufficiently large. For example, in Figure 3 (b), we see that

$\beta_{\max} = 1$ for $r = 1.9$ and $B = 10$, i.e., for these bounds the hybrid can consist completely of *PS*. The observed linear relationship suggests that the tightest bound in Algorithm 1 remains the same as the bounds (r, B) vary, but that the magnitude of manipulation changes with the utility function of the manipulating agent. Put differently, the structure of the most harmful manipulation is unchanged, but β_{\max} reacts to changes in the utility function.

Figure 3 (a) also suggests that the dependence of β_{\max} on B is inverse proportional. This makes sense, because large values of B make manipulations more attractive, such that a smaller β must be chosen to guarantee partial strategyproofness. However, β_{\max} is much less sensitive to increases in B than to decreases in r . In particular, for high values of B , the reduction of β_{\max} as B increases further becomes minimal (see plot in Figure 3 (b)). For extreme values like $r = 1.05$, it makes almost no difference whether $B = 1,000$ or $B = 10,000$. We find that $\beta_{\max} \approx 0.05$ in both cases (not shown in Figure 3).

9.2. Computational Complexity

Computing β_{\max} quickly becomes computationally hard. One reason for this is that the number of constraints to be checked grows exponentially in the number of goods and agents: there are $m!$ possible preference orderings, which yields $(m!)^n$ possible preference profiles. For each profile, the algorithm must iterate through all n agents and all $m!$ possible for each of them. Thus, the number of constraints to be checked is

$$O((m!)^n \cdot n \cdot m!) = O((m!)^{n+1} n).$$

Furthermore, for each of these constraints, the allocations of both components must be determined. While the algorithm we use to compute *PS* (see Algorithm 3 in Appendix H) has polynomial worst-case complexity $O(m^2(m+n))$, the algorithm we use to compute *RSD* (see Algorithm 2 in Appendix H) has exponential time complexity $O(n!m)$. This means that the overall runtime of Algorithm 1 applied to *RSD* and *PS* is

$$O((m!)^{n+1} n (m^2(m+n) + n!m)) = O((m!)^{n+2} (n!) n^2 m^3)$$

By exploiting anonymity and neutrality of *PS* and *RSD* and in settings with equal capacities, we reduce the number of constraints to

$$\binom{m! + n - 2}{n - 1},$$

improving performance in practice by a few orders of magnitude. The overall runtime then is

$$O\left(\binom{m! + n - 2}{n - 1} (m^2(m+n) + n!m)\right) = O\left(\binom{m! + n - 2}{n - 1} (n!) m^3\right)$$

Besides the large number of constraints, the second major bottleneck in the implementation was the complexity of *RSD*. However, to the best of our knowledge, no algorithm is known that can compute the non-deterministic allocation of *RSD* in worst-case polynomial time.

10. Conclusion

10.1. Summary

In this paper, we have studied the assignment problem, and our main contribution is a new paradigm we call *partial strategyproofness*, which is strategyproofness on a constrained subset of the utility space. The main constraint is that agents’ utilities must be bounded away from indifference. In contrast to *approximate strategyproofness*, we have not aimed for mechanisms that bound agents’ incentives to manipulate, but instead we have designed mechanisms that are “fully” strategyproof for the constrained subset of utility functions.

We have introduced *hybrid mechanisms*, which are convex combinations of existing mechanisms, with the goal to construct new mechanisms that are partially strategyproof and have good efficiency properties. In order to measure the efficiency improvements from using hybrids, we have developed the *imperfect dominance* concept, an extension of the canonical notions of dominance to mechanisms. Intuitively, we say that h imperfectly dominates f , if whenever h and f are comparable then h is the better choice. This efficiency concept allows us to make meaningful comparisons between hybrid mechanisms.

We have provided a large number of technical results (see Figure 1), but the most tangible one is without doubt Corollary 3, which is an instantiation of our theoretical results to two specific mechanisms. The corollary says that we can mix *RSD* and *PS*, such that the resulting hybrid mechanism is partially strategyproof and also strictly imperfectly dominates *RSD*. On a technical level, proving partial strategyproofness for the resulting hybrid required proving that *PS* is weakly less varying (on swaps) than *RSD*, i.e., that whenever *PS* changes the allocation upon a change of report (a swap), then so does *RSD*.

We have complemented our theoretical analysis with numerical results, showing that the maximal mixing factor can be surprisingly high for *PS* and *RSD*. For example, for a setting with 4 agents and 4 goods with unit capacity, if agents’ utilities are bounded away from indifference by 1.4 and are upper-bounded by 10, then we can mix *RSD* and *PS* at a ratio of 50/50, and the resulting hybrid is still strategyproof for the specified set of agents.

We believe that our theoretical and numerical results demonstrate that partial strategyproofness constitutes an interesting new mechanism design paradigm, enabling new kinds of trade-offs between efficiency and strategyproofness. In this way, our work sheds new light on the design space for assignment mechanisms, and shows that we do not need to settle for *RSD* and ex-post efficiency. In particular in small settings, where limit results do not apply, our hybrid mechanisms may even prove useful for practical applications.

10.2. Future Work and Open Research Questions

The work presented in this paper has raised a number of new research questions, some of which we plan on addressing in future work. First, we will tackle the tightness of the concept $URB(r, B)$. This will involve showing that hybrid mechanisms cannot be partially strategyproof if agents’ utilities are not bounded away from indifference. Note that the condition $URB(r, B)$ is not the tightest condition for which Theorem 1 and Corollary 1 can be proven. It is possible to replace $URB(r)$ by a weaker condition based on uniformly bounded constant

minimum increments. Nonetheless, we presented our findings for relative lower bounds for two reasons: first, the relative concept is much more intuitive to understand. Second, most popular mechanisms satisfy *weak invariance* (as defined in Hashimoto et al. (2013)), and we hypothesize that for weakly invariant mechanisms, the magnitude of manipulability depends on the relative rather than absolute utility differences. In terms of the sets of utilities for which partial strategyproofness can be guaranteed, we will numerically and analytically explore the advantages and disadvantages of different constraints on the utility space, including relative upper bounds and absolute lower bounds.

We have shown that previously existing efficiency concepts and dominance relations are not sufficient for the comparison of assignment mechanisms in general, and hybrid mechanisms in particular. For this reason, we have introduced the imperfect dominance concept, but this is only a first step towards developing new, more useful efficiency comparisons. In future work, we will continue studying new methods for efficiency comparisons, which will include using rank based and cardinal measures of welfare as well as different dominance relationships. We will also explore how partially strategyproof mechanisms compare to mechanisms on the efficient frontier as described in Erdil (2011).

On a conceptual level, our main contributions in this paper were *partial strategyproofness*, *uniformly relatively bounded utilities* and *imperfect dominance*. However, these concepts are not limited to assignment problems. In future work, we plan to apply these and similar concepts to *two-sided matching* problems and *social choice* problems. This may establish the generality of these concepts or show their limitations.

We also plan on continuing the formal analysis of hybrid mechanisms, including proving tighter analytical bounds for β as a function of the setting parameters. Based on our simulations, it looks like the maximum value for β is linearly increasing in r and inversely proportional to B , which is a relationship we would like to prove as well. Furthermore, an extension to the case of more than two component mechanisms may also be interesting. For the particular components *PS* and *RSD* we will consider non-uniform distributions over orderings and non-uniform eating speeds. Finally, to make hybrid mechanisms useful in practice, the speed at which the maximal mixing factor β_{\max} can be computed is essential. Thus, developing more efficient algorithms for finding β_{\max} is an interesting problem. At its core, however, this last challenge involves finding an algorithm for computing the stochastic allocation matrix for *RSD* in polynomial time, which has already proven to be a very difficult research problem.

References

- Abdulkadiroğlu, Atila, and Tayfun Sönmez. 1998. “Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems.” *Econometrica*, 66(3): 689–702.
- Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003a. “Ordinal Efficiency and Dominated Sets of Assignments.” *Journal of Economic Theory*, 112(1): 157–172.

- Abdulkadiroğlu, Atila, and Tayfun Sönmez.** 2003*b*. “School Choice: A Mechanism Design Approach.” *American Economic Review*, 93(3): 729–747.
- Abdulkadiroğlu, Atila, Parag A Pathak, and Alvin E. Roth.** 2005. “The New York City High School Match.” *American Economic Review*, 95(2): 364–367.
- Azevedo, Eduardo M., and Eric Budish.** 2012. “Strategyproofness in the Large as a Desideratum for Market Design.” In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*.
- Bogomolnaia, Anna, and Hervé Moulin.** 2001. “A New Solution to the Random Assignment Problem.” *Journal of Economic Theory*, 100(2): 295–328.
- Budish, Eric.** 2011. “The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes.” *Journal of Political Economy*, 119(6): 1061–1103.
- Budish, Eric.** 2012. “Matching “versus” Mechanism Design.” *ACM SIGecom Exchanges*, 11(2): 4–15.
- Budish, Eric, and Estelle Cantillon.** 2012. “The Multi-Unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard.” *American Economic Review*, 102(5): 2237–2271.
- Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom.** 2013. “Designing Random Allocation Mechanisms: Theory and Applications.” Forthcoming in *American Economic Review*.
- Calsamiglia, Caterina, and Antonio Miralles.** 2012. “All About Priorities: No School Choice Under the Presence of Bad Schools.” Working Papers 631, Barcelona Graduate School of Economics.
- Carroll, Gabriel.** 2011*a*. “A Quantitative Approach to Incentives: Application to Voting Rules.” Working Paper, Department of Economics, MIT.
- Carroll, Gabriel.** 2011*b*. “On Mechanisms Eliciting Ordinal Preferences.” Working Paper, Department of Economics, MIT.
- Carroll, Gabriel.** 2012. “When Are Local Incentive Constraints Sufficient?” *Econometrica*, 80(2): 661–686.
- Dütting, Paul, Felix Fischer, Pichayut Jirapinyo, John K. Lai, Benjamin Lubin, and David C. Parkes.** 2012. “Payment Rules Through Discriminant-based Classifiers.” In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC)*.
- Erdil, Aytek.** 2011. “Strategy-proof Stochastic Assignment.” Working Paper, Faculty of Economics, University of Cambridge.

- Ergin, Haluk, and Tayfun Sonmez.** 2006. "Games of School Choice Under the Boston Mechanism." *Journal of Public Economics*, 90(1-2): 215–237.
- Featherstone, Clayton R.** 2011. "A Rank-based Refinement of Ordinal Efficiency and a new (but Familiar) Class of Ordinal Assignment Mechanisms." Working Paper, The Wharton School, University of Pennsylvania.
- Hashimoto, Tadashi, Daisuke Hirata, Onur Kesten, Morimitsu Kurino, and M. Utku Ünver.** 2013. "Two Axiomatic Approaches to the Probabilistic Serial Mechanism." Forthcoming in *Theoretical Economics*.
- Hatfield, J.** 2009. "Strategy-proof, Efficient, and Nonbossy Quota Allocations." *Social Choice and Welfare*, 33(3): 505–515.
- Heo, Eun Jeong, and Özgür Yilmaz.** 2011. "A Characterization of the Extended Serial Correspondence." Working Paper, Department of Economics, University of Rochester.
- Hylland, Aanund, and Richard Zeckhauser.** 1979. "The Efficient Allocation of Individuals to Positions." *The Journal of Political Economy*, 87(2): 293–314.
- Katta, Akshay-Kumar, and Jay Sethuraman.** 2006. "A Solution to the Random Assignment Problem on the Full Preference Domain." *Journal of Economic Theory*, 131(1): 231–250.
- Kesten, Onur, and Özgün Ekici.** 2012. "An Equilibrium Analysis of the Probabilistic Serial Mechanism." Working Paper, Özeygin University, Istanbul,.
- Kojima, Fuhito, and Mihai Manea.** 2010. "Incentives in the Probabilistic Serial Mechanism." *Journal of Economic Theory*, 145(1): 106–123.
- Lubin, Benjamin, and David C. Parkes.** 2012. "Approximate Strategyproofness." *Current Science*, 103(9): 1021–1032.
- Niederle, Muriel, Alvin E. Roth, and Tayfun Sönmez.** 2008. "Matching and Market Design." In *The New Palgrave Dictionary of Economics*. Palgrave Macmillan.
- Pathak, Parag A., and Tayfun Sönmez.** 2013. "School Admissions Reform in Chicago and England: Comparing Mechanisms by Their Vulnerability to Manipulation." *American Economic Review*, 103(1): 80–106.
- von Neumann, John.** 1953. "Contributions to the Theory of Games." , ed. Harold W. Kuhn Albert W. Tucker Vol. 2, Chapter A Certain Zero-sum Two-person Game Equivalent to the Optimal Assignment Problem. Princeton University Press, Princeton, New Jersey.
- von Neumann, John, and Oskar Morgenstern.** 1944. *Theory of Games and Economic Behavior*. Princeton University Press.
- Zhou, Lin.** 1990. "On a Conjecture by Gale about One-sided Matching Problems." *Journal of Economic Theory*, 52(1): 123–135.

Appendices

A. Proofs from Section 4: Hybrid Mechanisms

A.1. Proofs from Section 4.1: Construction of Hybrid Assignment Mechanisms

Proof of Proposition 1. The convex combination of allocations is an allocation.

By induction it suffices to prove the result for two allocations. Let $X, Y \in \mathcal{X}$ be allocations and $Z = (1 - \beta)X + \beta Y$ for some $\beta \in [0, 1]$. Z is again a matrix with n rows and m columns. We have $z_{i,j} = (1 - \beta)x_{i,j} + \beta y_{i,j}$ for all $i \in N, j \in M$. Then for any $i \in N$ and $j \in M$, respectively, we have

$$\sum_{i \in N} z_{i,j} = \sum_{i \in N} (1 - \beta)x_{i,j} + \beta y_{i,j} = (1 - \beta) \sum_{i \in N} x_{i,j} + \beta \sum_{i \in N} y_{i,j} = (1 - \beta)q_j + \beta q_j = q_j,$$

and

$$\sum_{j \in M} z_{i,j} = \sum_{j \in M} (1 - \beta)x_{i,j} + \beta y_{i,j} = (1 - \beta) \sum_{j \in M} x_{i,j} + \beta \sum_{j \in M} y_{i,j} = 1 - \beta + \beta = 1,$$

which proves the result. \square

B. Proofs from Section 5: Existence and Computability of Non-trivial Hybrid Mechanisms

B.1. Proofs from Section 5.1: Existence Result

Proof of Theorem 1. Fix a setting (N, M, \mathbf{q}) . The following statements are equivalent.

1. f is a strategyproof mechanism in the setting (N, M, \mathbf{q}) .
2. For all $B \geq r^{m-1}, r > 1$ and any mechanism $g \ll_{\text{swap}} f$ there exists $\beta > 0$, such that $h_\beta(f, g)$ is strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .

Show “ \Rightarrow ” Fix reports from the other agents $t_{-i} \in T^{n-1}$. We use the abbreviated form $f(t)$ to denote $f(t_i, t_{-i})_i$. Suppose that f is SP and that a setting (N, M, \mathbf{q}) , $g \ll_{\text{swap}} f$ and $B \geq r^{m-1}, r \geq r > 1$ are given. We need to show that there exists $\beta > 0$ such that

$$\langle u, h_\beta(t) - h_\beta(t') \rangle \geq 0 \tag{9}$$

holds for any $t, t' \in T$ and any $u \in t \cap \text{URB}(r, B, m)$.

First we require two more definitions: the *canonical transition* between two types and the *minimum change* of f under swaps.

Define as the *canonical transition* from t to t' the sequence $\text{Can}(t, t') = (t_0 = t, t_1, \dots, t_{k-1}, t_k = t')$ of types in the following way:

1. Identify the first ranked good in t' .
2. Identify the sequence of swaps needed to bring this good to first position, beginning at the preference order of t .
3. Identify the second ranked good in t' , and identify the swaps needed to bring it to second position, beginning at the preference order created by all previous swaps.
4. Repeat this procedure for all positions in t' .
5. Define $\text{Can}(t, t')$ as the sequence of types generated by this sequence of swaps.

Observe that $t_l \in N_{t_{l+1}}$ and $t_{l+1} \in N_{t_l}$ for all $l \in \{0, \dots, k-1\}$, i.e., consecutive types in $\text{Can}(t, t')$ are neighbors. Second, note that every pair of goods x, y is swapped at most once. Therefore, if t is the true type of an agent, any swap transition will change the order of two goods from the true order to the wrong order.

Let $t \in T$ be a type and $t' \in N_t$ a type in the neighborhood of t . Let x, y be the goods that get swapped in the transition from t to t' . f is SP, hence Lemma 1 yields that $f(t)$ and $f(t')$ differ only in the probability for x and y , and $(f(t) - f(t'))_y = -(f(t) - f(t'))_x$. Define

$$\epsilon_f = \min\{\epsilon_{(f,t,t')} : \epsilon_{(f,t,t')} = (f(t) - f(t'))_x \text{ for some } t \in T, t' \in N_t, \epsilon_{(f,t,t')} > 0, x \in M\}.$$

The above argument guarantees that $\epsilon_f > 0$.¹⁰

Fix t, t' , then $g(t) \neq g(t')$. Otherwise $\langle u, h_\beta(t) - h_\beta(t') \rangle = \langle u, f(t) - f(t') \rangle \geq 0$ for any β . If $g(t) \neq g(t')$, there must be at least one $l \in \{0, \dots, k-1\}$ such that $g(t_{l+1}) \neq g(t_l)$. Then $f(t_{l+1}) \neq f(t_l)$, because g is weakly less varying on swaps than f by assumption. Let x, y be the two goods swapped in the transition from t_l to t_{l+1} , changing $x \succ_l y$ to $y \succ_{l+1} x$, say. Then for any $u \in t$ we have

$$\begin{aligned} \langle u, f(t_l) - f(t_{l+1}) \rangle &= u(x)(f(t_l) - f(t_{l+1}))_x + u(y)(f(t_l) - f(t_{l+1}))_y \\ &= (u(x) - u(y))(f(t_l) - f(t_{l+1}))_x \\ &\geq (u(x) - u(y))\epsilon_f. \end{aligned}$$

Here the last inequality follows because t_l and t_{l+1} are consecutive elements of the canonical transition from t to t' . Hence, $x \succ_t y$ in the true preference order of the agent, which implies $u(x) > u(y)$.

Now, for some $u \in t \cap \text{URB}(r, B, m)$ we have $u(x) \geq ru(y)$ and $u(y) \geq 1$. It follows that

$$\langle u, f(t_l) - f(t_{l+1}) \rangle \geq (r-1)\epsilon_f.$$

None of the other swaps in $\text{Can}(t, t')$ will increase the expected utility by the same argument. Thus

$$\langle u, f(t) - f(t') \rangle = \sum_{l=0}^{k-1} \langle u, f(t_l) - f(t_{l+1}) \rangle \geq (r-1)\epsilon_f > 0. \quad (10)$$

¹⁰Note that ϵ_f may depend on the setting, but is independent of g, r, B .

This establishes a lower bound for the loss in case a misreport changes the allocation. Now we upper-bound the potential gain from manipulating g . In the worst-case, an agent of type t can attain its most preferred good with certainty under g by reporting type t' , but would receive its least preferred good when reporting t . Then if $u \in t \cap \text{URB}(r, B, m)$, we have

$$\langle u, g(t') - g(t) \rangle \leq B - 1,$$

or equivalently

$$\langle u, g(t) - g(t') \rangle \geq 1 - B. \quad (11)$$

Choose $\beta = \frac{\epsilon_f(r-1)}{B-1+\epsilon_f(r-1)} > 0$, then $1 - \beta = \frac{B-1}{B-1+\epsilon_f(r-1)}$ and use (10) and (11) to get

$$\begin{aligned} \langle u, h_\beta(t) - h_\beta(t') \rangle &= (1 - \beta) \langle u, f(t) - f(t') \rangle + \beta \langle u, g(t) - g(t') \rangle \\ &\geq (1 - \beta) \epsilon_f(r - 1) + \beta(1 - B) \\ &= \epsilon_f(r - 1) \left((1 - \beta) + \frac{1 - B}{B - 1 + \epsilon_f(r - 1)} \right) = 0 \end{aligned}$$

f and g still depend on the report \succ_{-i} from all other agents. However, the upper bound for g was independent of this report, and the lower bound depends on \succ_{-i} through the existence of a $\epsilon_f > 0$. But since f is strategyproof regardless of \succ_{-i} , we can repeat this argument for any \succ_{-i} and find a $\epsilon_{f, \succ_{-i}} > 0$ each time. There are only finitely many possibilities for \succ_{-i} , hence we can choose the smallest and obtain a β for which inequality (9) is satisfied for any $t, t' \in T$ and any $u \in t \cap \text{URB}(r, B, m)$.

Show “ \Leftarrow ” By contradiction: suppose f is not strategyproof, then there exist types $t_{-i} \in T^{n-1}$, $t, t' \in T$, $u \in t$ (again using abbreviated notation) such that

$$\langle u, f(t) - f(t') \rangle < 0. \quad (12)$$

Multiplying u by a scalar and adding or subtracting a scalar multiple of the unit vector, we construct a $u' \in t$ for which inequality (12) holds as well, but with $\min u' = 1$. Because indifference is not permitted, we can define

$$r_{u'} = \min_{x, y \in M, x \text{ predecessor of } y} \frac{u'(x)}{u'(y)} > 1, B_{u'} = \frac{\max_{x \in M} u'(x)}{\min_{y \in M} u'(y)} \geq r_{u'}^{m-1}.$$

Then we choose $g = f$, such that $g \ll_{\text{swap}} f$ and $h_\beta = f$ for any $\beta \in [0, 1]$. But then $h_\beta = f$ is not strategyproof for $u' \in t \cap \text{URB}(r_{u'}, B_{u'}, m)$ for any $\beta > 0$, a contradiction. □

Proof of Lemma 1. For any $t_{-i} \in T^{n-1}$, let $t_i \in T$, $t'_i \in N_t$, and let $x, y \in M$ be the goods that change position, such that $x \succ_i y$ and $y \succ'_i x$. Then for any strategyproof mechanism f we get

$$(f(t'_i, t_{-i})_i - f(t_i, t_{-i})_i)(x) = (f(t_i, t_{-i})_i - f(t'_i, t_{-i})_i)(y),$$

and for all goods $z \neq x, y$ we have

$$(f(t'_i, t_{-i})_i - f(t_i, t_{-i})_i)(z) = 0.$$

We use the abbreviated notation $f(t)$ for $f(t_i, t_{-i})_i$. Fix reports from other agents $t_{-i} \in T^{n-1}$. The proof works in 3 steps: first, we show (1) for goods ranked above x in t (and hence t' as well). Second, we show (1). The third step is to show (1) for all goods ranked below y in t . Recall that $\text{rank}_t(j)$ of a good j at t is the number of goods that a type t agent weakly prefers to j , i.e., its first choice has rank 1, its second choice has rank 2, etc.

Higher-ranking Goods Consider a good $a \neq x, y$, for which $a \succ_t x$ and $a \succ_{t'} y$. Suppose $(f(t))_a \neq (f(t'))_a$, i.e., this good receives different probability under the different reports. Assume that a is the highest ranking good for which this is the case. Without loss of generality¹¹ suppose

$$(f(t') - f(t))(a) =: \Delta(a) > 0.$$

Then let u_ϵ be a utility function in t with

$$u_\epsilon(j) = \begin{cases} 1 + \epsilon(\text{rank}_t(a) - \text{rank}_t(j)), & \text{if } j \succ_t a, \\ 1, & \text{if } j = a \\ \epsilon(m - \text{rank}_t(j)), & \text{if } a \succ_t j. \end{cases}$$

Since the probabilities for goods j with $j \succ_t a$ do not change, we get

$$\langle u_\epsilon, f(t') - f(t) \rangle \geq \Delta(a) - m^2 \epsilon > 0$$

for sufficiently small $\epsilon > 0$. This means that an agent of type t with utility u_ϵ has an incentive to misreport its type as t' , a contradiction to strategyproofness.

Inverse Changes for x and y By the first step, (1) holds for any good ranked above x and y . So we can assume without loss of generality that x and y are the goods most preferred under t and t' , respectively. Suppose (1) did not hold. Then without loss of generality we have

$$(f(t') - f(t))(x) - (f(t) - f(t'))(y) := \Delta(x) + \Delta(y) > 0.$$

Let u_ϵ be a utility function in t with

$$u_\epsilon(j) = \begin{cases} 1 + \epsilon, & \text{if } j = x, \\ 1, & \text{if } j = y \\ \epsilon(m - \text{rank}_t(j)), & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \langle u_\epsilon, f(t') - f(t) \rangle &\geq \Delta(x)(1 + \epsilon) + \Delta(y) - m^2 \epsilon \\ &> \Delta(x) + \Delta(y) - m^2 \epsilon > 0 \end{aligned}$$

for sufficiently small $\epsilon > 0$. This positive incentive to misreport again contradicts strategyproofness.

¹¹Otherwise consider an agent of type t' .

Lower-ranking Goods Consider a good $b \neq x, y$, for which $y \succ_t b$ and $x \succ_{t'} b$. Suppose $(f(t))_b \neq (f(t'))_b$ and that b is the highest ranking good for which this is the case. Without loss of generality suppose

$$(f(t') - f(t))(b) = \Delta(b) > 0.$$

Let $u_\epsilon \in t$ be the utility function constructed in the first step but with b taking the place of a . For the incentives to misreport we get

$$\begin{aligned} \langle u_\epsilon, f(t') - f(t) \rangle &> \Delta(x)u_\epsilon(x) + \Delta(y)u_\epsilon(y) + \Delta(b) - m^2\epsilon \\ &\geq \Delta(x)\epsilon + \Delta(b) - m^2\epsilon \\ &= \Delta(b) - \epsilon(m^2 - \Delta(x)) > 0 \end{aligned}$$

for sufficiently small $\epsilon > 0$. Like in the previous steps, this is a contradiction. □

B.2. Proofs from Section 5.2: Computability of the Maximal Mixing Factor

Proof of Proposition 2. Fix bounds (r, B) and a setting (N, M, \mathbf{q}) . For type $t \in T$ suppose that

$$\succ: j_1 > \dots > j_m$$

is the corresponding preference order. Then let

$$\eta(t, r, B, m) = \left\{ u \in t : \exists k \in \{1, \dots, m\} \text{ with } u(j_l) = \begin{cases} r^{m-l}, & \text{if } l \geq m - k + 1, \\ Br^{1-l}, & \text{if } l \leq m - k \end{cases} \right\}.$$

Then for any mechanism h the following statements are equivalent.

1. h is strategyproof in $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .
2. For any agent i , any $t_i, t'_i \in T$, any $t_{-i} \in T^{n-1}$, and any $u_i \in \eta(t_i, r, B, m)$ we have

$$\langle u_i, h(t_i, t_{-i})_i - h(t'_i, t_{-i})_i \rangle \geq 0.$$

For the sake of brevity, we drop the index i throughout the proof. The “only-if” part is clear, because $\eta(t, r, B, m) \subset \text{URB}(r, B, m)$, and if h is partially strategyproof on $\text{URB}(r, B, m)$, inequality (7) is satisfied.

For the “if” part we use that if h is $\{u, u'\}$ -strategyproof for $u, u' \in t$, then h is $\{(1-\alpha)u - \alpha u' : 0 \leq \alpha \leq 1\}$ -strategyproof. This is (4) in Section 4.1.

Now we show that $\text{convex}(\eta(t, r, B, m)) = t \cap \text{URB}(r, B, m)$: $\eta(t, r, B, m) \subset t$ is clear. Because of the definition of the $u \in \eta(t, r, B, m)$, any of them satisfies $\text{URB}(r, B)$, hence $\eta(t, r, B, m) \subset t \cap \text{URB}(r, B, m)$.

We need to show that any $u \in t \cap \text{URB}(r, B, m)$ can be represented as a convex combination of elements in $\eta(t, r, B, m)$. For $m = 2$, suppose $x > y$ for agents of type t . Then $u(x) \in [ru(y), B]$

and $\eta(t, r, B, m) = \{(u^+(x) = B, u^+(y) = 1), (u^-(x) = r, u^-(y) = 1)\}$. Choose $\alpha = \frac{u(x)-r}{B-r}$, then $(1 - \alpha)u^- + \alpha u^+ = u$.

For more (m) goods, let $m = m' + 1$ and $u = (u_1, \dots, u_{m'}, u_m)$. By the induction hypothesis (using $B' = \frac{B}{r}$), we can construct $u^- = u = (u_1, \dots, u_{m'}, ru_{m'})$ and $u^+ = u = (u_1, \dots, u_{m'}, B)$: for u^- , the coefficient of $(1, r, \dots, r^{m'-1})$ from the m' -dimensional construction becomes the coefficient of $(1, r, \dots, r^{m-1})$ in the m -dimensional construction. For u^+ use this coefficient for $(1, r, \dots, r^{m'-1}, B)$. Then u is a convex combination of u^- and u^+ . \square

Proof of Proposition 3. If f and g are computable, Algorithm 1 is correct and complete, i.e., the algorithm terminates on all valid inputs, and for a setting (N, M, \mathbf{q}) , mechanisms f, g with f SP, $g \ll_{\text{swap}} f$, and bounds (r, B) , Algorithm 1 finds a β_{\max} such that $h_{\beta_{\max}}$ is partially strategyproof on $\text{URB}(r, B, m)$ in the setting (N, M, \mathbf{q}) .

The algorithm can only reduce the value of β in every step of the loops. It considers every agent $i \in N$ and every possible report $t_{-i} \in T^{n-1}$ from the other agents. We drop the index i .

The inner loops iterate through every combination of t types that an agent could have and every possible report t' from that agent. It also considers every utility $u \in \eta(t, r, B, m)$ in the possible true types. By Proposition 2 the mechanism h_β is partially strategyproof on $\text{URB}(r, B, m)$ if (7) holds for h_β for each of these combinations. Let $\Delta(u, f, t, t') = \langle u, f(t) - f(t') \rangle$ and $\Delta(u, g, t, t') = \langle u, g(t') - g(t) \rangle$ and note that $\Delta(u, \cdot, t, t') = -\Delta(u, \cdot, t', t)$. There are two cases:

Suppose $\Delta(u, f, t, t') \geq \Delta(u, g, t', t)$ If $\Delta(u, f, t, t') = \Delta(u, g, t', t)$, then $\langle u, h_\beta(t) - h_\beta(t') \rangle = 0$ for any $\beta \in [0, 1]$. If $\Delta(u, f, t, t') > \Delta(u, g, t', t)$, then condition (7) becomes

$$\begin{aligned} \langle u, h_\beta(t) - h_\beta(t') \rangle &\geq 0 \\ \Leftrightarrow (1 - \beta)\Delta(u, f, t, t') + \beta\Delta(u, g, t, t') &\geq 0 \\ \Leftrightarrow \beta(\Delta(u, f, t, t') - \Delta(u, g, t', t)) &\geq -\Delta(u, f, t, t') \\ \Leftrightarrow \beta &\geq \frac{-\Delta(u, f, t, t')}{\Delta(u, f, t, t') - \Delta(u, g, t', t)}, \end{aligned} \quad (13)$$

where the last equivalence follows because $\Delta(u, f, t, t') > \Delta(u, g, t', t)$. f is SP, so $-\Delta(u, f, t, t') \leq 0$. But if the right side of (13) is non-positive, the condition holds for any $\beta \geq 0$.

Suppose $\Delta(u, f, t, t') < \Delta(u, g, t', t)$ Then by the same argument as in the previous case we get

$$\begin{aligned} \langle u, h_\beta(t) - h_\beta(t') \rangle &\geq 0 \\ \Leftrightarrow \beta(\Delta(u, f, t, t') - \Delta(u, g, t', t)) &\geq -\Delta(u, f, t, t') \\ \Leftrightarrow \beta &\leq \frac{\Delta(u, f, t, t')}{\Delta(u, g, t', t) - \Delta(u, f, t, t')}, \end{aligned} \quad (14)$$

where the direction of the last inequality changes because $\Delta(u, f, t, t') - \Delta(u, g, t', t) < 0$.

Algorithm 3 always reduces β if constraint (14) is violated. So after all loops, (7) from Proposition 2 is satisfied for h_β .

On the other hand, any larger $\beta' > \beta$ would violate constraint (14) for some t, t' and $u \in \eta(t, r, B, m)$. Then $h_{\beta'}$ violates constraint (7) for the same $t, t', u \in \eta(t, r, B, m)$, so h_β is not partially strategyproof on $\text{URB}(r, B, m)$ by Proposition 2. This proves correctness.

f and g are computable by assumption. For a given setting (N, M, \mathbf{q}) , the sets T^{n-1}, T and $\eta(T, r, B, m) = \bigcup_{t \in T} \eta(t, r, B, m)$ are finite. Hence, the nested loops run finitely many times. This shows completeness. \square

C. Proofs from Section 6: Efficiency of Hybrid Mechanisms

Proof of Proposition 4. For $X, Y \in \mathcal{X}$, X ex-post efficient, let $Z = (1 - \beta)X + \beta Y$ with $\beta \in (0, 1]$. Then

1. Y ex-post efficient $\Rightarrow Z$ ex-post efficient,
2. Y (strictly) ordinally dominates $X \Rightarrow Z$ (strictly) ordinally dominates X ,
3. Y (strictly) rank dominates $X \Rightarrow Z$ (strictly) rank dominates X .

Show 4.(1) If X and Y are both ex-post efficient, each has a lottery representation over ex-post efficient deterministic allocation. Then the β -convex combination of these lotteries is again a lottery over ex-post efficient deterministic allocations and implements $(1 - \beta)X + \beta Y$.

Show 4.(2) If $Y \geq_{OD} X$, then for all $i \in N$ we have $Y_i \geq_i X_i$, i.e., for all $j \in M$

$$\sum_{j': j' \geq_i j} y_{i, j'} \geq \sum_{j': j' \geq_i j} x_{i, j'}. \quad (15)$$

Then

$$\begin{aligned} \sum_{j': j' \geq_i j} ((1 - \beta)Y + \beta X)_{i, j'} &= \sum_{j': j' \geq_i j} (1 - \beta)Y_{i, j'} + \beta X_{i, j'} \\ &= (1 - \beta) \sum_{j': j' \geq_i j} y_{i, j'} + \beta \sum_{j': j' \geq_i j} x_{i, j'} \\ &\geq (1 - \beta) \sum_{j': j' \geq_i j} x_{i, j'} + \beta \sum_{j': j' \geq_i j} x_{i, j'} = \sum_{j': j' \geq_i j} x_{i, j'}, \end{aligned} \quad (16)$$

or $((1 - \beta)X + \beta Y) \geq_{OD} X$. If $Y >_{OD} X$, inequality (15) is strict for some agent i and some good j . Then the estimate (16) for i and j is also strict with $\beta > 0$. Hence, $((1 - \beta)X + \beta Y) >_{OD} X$.

Show 4.(3) Similarly, if $Y \geq_{RD} X$, then for all $l \in \{1, \dots, m\}$ we have

$$\sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} y_{i, j} \geq \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} x_{i, j}. \quad (17)$$

Then

$$\begin{aligned}
\sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} ((1 - \beta)X + \beta Y)_{i,j} &= \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} (1 - \beta)x_{i,j} + \beta y_{i,j} \\
&= (1 - \beta) \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} x_{i,j} \\
&\quad + \beta \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} y_{i,j} \\
&\geq \sum_{i \in N} \sum_{j \in M: \text{rank}_i(j) \leq l} y_{i,j}, \tag{18}
\end{aligned}$$

or $((1 - \beta)X + \beta Y) \geq_{RD} X$. If $Y >_{RD} X$, inequality (17) is strict for some rank l . Then the estimate (18) for this l is also strict with $\beta > 0$. Hence, $((1 - \beta)X + \beta Y) >_{RD} X$. \square

C.1. Proofs from Section 6.2: Efficiency Comparison of Hybrid Mechanisms

Proof of Proposition 8. For mechanisms f, g , let $h_\beta = (1 - \beta)f + \beta g$ be a hybrid with $\beta \in (0, 1]$. Then

1. $g \geq_{IOD} f \Rightarrow h \geq_{IOD} f$ and $g \geq_{IRD} f \Rightarrow h \geq_{IRD} f$,
2. $g >_{IOD} f \Rightarrow h >_{IOD} f$ and $g >_{IRD} f \Rightarrow h >_{IRD} f$.

Show 1.: By contradiction: suppose that $h \geq_{IOD} f$ does not hold. Then for some type profile $\tilde{\mathbf{t}}$, the allocation $f(\tilde{\mathbf{t}})$ strictly ordinally dominates the allocation $h(\tilde{\mathbf{t}})$, i.e., for all agents i and all goods j

$$\sum_{j' \geq i j} f(\tilde{\mathbf{t}})_{i,j'} \geq \sum_{j' \geq i j} h(\tilde{\mathbf{t}})_{i,j'} = \sum_{j' \geq i j} (1 - \beta)f(\tilde{\mathbf{t}})_{i,j'} + \beta g(\tilde{\mathbf{t}})_{i,j'}, \tag{19}$$

and there exists an agent \tilde{i} and a good \tilde{j} such that

$$\sum_{j' \geq \tilde{i} \tilde{j}} f(\tilde{\mathbf{t}})_{\tilde{i},j'} > \sum_{j' \geq \tilde{i} \tilde{j}} h(\tilde{\mathbf{t}})_{\tilde{i},j'} = \sum_{j' \geq \tilde{i} \tilde{j}} (1 - \beta)f(\tilde{\mathbf{t}})_{\tilde{i},j'} + \beta g(\tilde{\mathbf{t}})_{\tilde{i},j'} \tag{20}$$

(19) implies that

$$\sum_{j' \geq i j} \beta f(\tilde{\mathbf{t}})_{i,j'} \geq \sum_{j' \geq i j} \beta g(\tilde{\mathbf{t}})_{i,j'},$$

i.e., $f(\tilde{\mathbf{t}})$ weakly ordinally dominates $g(\tilde{\mathbf{t}})$, and (20) implies that this dominance is actually strict. But by assumption $g \geq_{IOD} f$, hence $\neg(\exists \mathbf{t} : f(\mathbf{t}) > g(\mathbf{t}))$, a contradiction.

For rank dominance, the proof is analogous: suppose that $h \geq_{IRD} f$ does not hold, then for every rank r we get

$$\sum_{i \in M} \sum_{j: \text{rank}_i(j) \leq r} f(\tilde{\mathbf{t}})_{i,j} \geq \sum_{i \in M} \sum_{j: \text{rank}_i(j) \leq r} h(\tilde{\mathbf{t}})_{i,j},$$

and for some rank \tilde{r}

$$\sum_{i \in M} \sum_{j: \text{rank}_i(j) \leq \tilde{r}} f(\tilde{\mathbf{t}})_{i,j} > \sum_{i \in M} \sum_{j: \text{rank}_i(j) \leq \tilde{r}} h(\tilde{\mathbf{t}})_{i,j}.$$

Replacing h by the convex combination, we get that $f(\tilde{\mathbf{t}})$ strictly rank dominates $g(\tilde{\mathbf{t}})$, a contradiction.

Show 2.: From 1. we know that the h weakly imperfectly dominates f . If g strictly imperfectly dominates f , then there exists a type profile $\tilde{\mathbf{t}}$ at which $g(\tilde{\mathbf{t}})$ strictly dominates $f(\tilde{\mathbf{t}})$. The strict versions of Proposition 4, 2. and 3. yield that $h(\tilde{\mathbf{t}})$ strictly dominates $f(\tilde{\mathbf{t}})$ as well. This implies strict imperfect dominance. □

D. Proofs from Section 7: Hierarchies of Efficiency and Manipulability

D.1. Proofs from Section 7.2: Manipulability Hierarchy

Proof of Proposition 10. Let f be strategyproof and $g \ll_{\text{swap}} f$, and let g be manipulable for some setting (M, N, \mathbf{q}) . Then for $0 \leq \beta < \beta' \leq 1$, $h_{\beta'}$ is intensely and strongly more manipulable than h_{β} .

Suppose, h_{β} is manipulable, then there exist $t_{-i} \in T^{n-1}, t, t' \in T, u \in t$ such that

$$\langle u, h_{\beta}(t) - h_{\beta}(t') \rangle < 0,$$

where we use abbreviate $h(t_i, t_{-i})_i$ as $h(t)$. Since f is strategyproof and $g \ll_{\text{swap}} f$, we get $g \ll f$ from Lemma 2. Therefore,

$$\langle u, f(t) - f(t') \rangle > 0$$

and

$$\langle u, g(t) - g(t') \rangle < 0.$$

With $\beta' > \beta$, this implies

$$\begin{aligned} \langle u, h_{\beta}(t) - h_{\beta}(t') \rangle &= (1 - \beta) \langle u, f(t) - f(t') \rangle + \beta \langle u, g(t) - g(t') \rangle \\ &> (1 - \beta') \langle u, f(t) - f(t') \rangle + \beta' \langle u, g(t) - g(t') \rangle \\ &= \langle u, h_{\beta'}(t) - h_{\beta'}(t') \rangle. \end{aligned}$$

This proves (23) with $t^* = t'$.

We use Lemma 3 to determine values $B \geq r^{m-1}, r > 1$ such that Algorithm 1 selects $\beta_{\max} = \beta$. Then there exist types $t_i, t'_i \in T$, some $t_{-i} \in T^{n-1}$, and a utility $u \in \eta(t_i, r, B, m)$ (using r and B) for which the constraint

$$\langle u, h_{\beta}(t_i) - h_{\beta}(t'_i) \rangle \geq 0$$

is tight, i.e., it is an equality. If $\langle u, f(t_i) - f(t'_i) \rangle = 0$, any β would have been allowed, hence $\langle u, f(t_i) - f(t'_i) \rangle > 0$ and $\langle u, g(t_i) - g(t'_i) \rangle < 0$. Since the constraint is tight, any larger choice β' would result in a mechanism $h_{\beta'}$ that is manipulable by an agent of type t_i with utility u . \square

Lemma 2. *If f is strategyproof, then $g \ll_{\text{swap}} f \Leftrightarrow g \ll f$.*

Proof. Let f be strategyproof and $g \ll_{\text{swap}} f$. Suppose, $g \ll f$ does not hold, then there exist types $t_{-i} \in T^{n-1}, t, t' \in T$ such that $g(t) \neq g(t')$, but $f(t) = f(t')$. Let $\text{Can}(t, t')$ be the canonical transition from t to t' as defined in the proof of Theorem 1. From $g(t) \neq g(t')$ we know that there exist types t_l, t_{l+1} in $\text{Can}(t, t')$ such that $t_l \in N_{t_{l+1}}$ and $g(t_l) \neq g(t_{l+1})$. $g \ll_{\text{swap}} f$ holds by assumption, thus $f(t_l) \neq f(t_{l+1})$. Suppose that $x >_{t_l} y$ and $y >_{t_{l+1}} x$. From Lemma 1 we know that $f(t')_x > f(t)_x$. Due to the construction of the canonical transition: x will only change positions with goods ranked higher than x . Thus, the probability of getting x can only increase further or remain constant. We get $f(t')_x \geq f(t_{l+1})_x > f(t_l)_x \geq f(t)_x$. This is a contradiction.

The other direction of the proof is trivial. \square

Lemma 3. *Given a setting, if f is SP, g not SP and $g \ll_{\text{swap}} f$, then for any $\beta \in (0, 1)$ there exist $B \geq r^{m-1}, r \geq 1$ such that $\beta = \beta_{\max}$ from Algorithm 1 applied to the setting (N, M, \mathbf{q}) , mechanisms f, g , and bounds (r, B) .*

Proof. We show that $\beta_{\max} : (1, \infty)^2 \rightarrow [0, 1]$ is a continuous mapping. Because $\lim_{(r, B) \rightarrow (1, 1)} \beta_{\max}(r, B) = 1$ and $\lim_{(r, B) \rightarrow (1, \infty)} \beta_{\max}(r, B) = 0$, the interim value theorem then yields that such r, B must exist.

Let $\delta > 0$ be small. Define $r' = (1 \pm \delta)r, B' = (1 \pm \delta)B$. For some $u \in \eta(t, r, B, m)$ let u' be the respective utility function in $\eta(t, r', B', m)$, i.e., the set constructed with the adjusted r' and B' . Then

$$\begin{aligned} & |\langle u', h(t) - h(t') \rangle - \langle u, h(t) - h(t') \rangle| \\ & \leq \sum_{j=1}^m |1 - (1 \pm \delta)^j| u_j \leq \delta C, \end{aligned}$$

for some positive constant C . That means, incentives are continuous in r and B , then so is β_{\max} . \square

E. Proofs from Section 8: Hybrid Instantiations: Mixing *RSD* with *PS* and *RV*

E.1. Proofs from Section 8.1: Mixing *RSD* and *PS*

Proof of Theorem 2. For any setting (N, M, \mathbf{q}) , *PS* is weakly less varying on swaps than *RSD*, i.e., $PS \ll_{\text{swap}} RSD$.

Suppose, n agents compete for $m = m_a + 2 + m_b$ goods with capacities given by \mathbf{q} , and let $M = \{a_1, \dots, a_{m_a}, x, y, b_1, \dots, b_{m_b}\}$. Agent 1 is considering the two type reports

$$t_1 : a_1 >_1 \dots >_1 a_{m_a} >_1 x >_1 y >_1 b_1 >_1 \dots >_1 b_{m_b}$$

and

$$t'_1 : a_1 >'_1 \dots >'_1 a_{m_a} >'_1 y >'_1 x >'_1 b_1 >'_1 \dots >'_1 b_{m_b},$$

where the positions of x and y are reversed in the second report. The reports of the other agents are fixed and given by $>_{-1}$.

Further suppose that with reports $(>_1, >_{-1})$, the goods were exhausted at times $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m = 1$ under PS . Re-label the goods as j_1, \dots, j_m in increasing order of the times at which they were exhausted. If two goods were exhausted at the same time, relabel them in arbitrary order. Denote by τ_x and τ_y the times at which x and y were exhausted, respectively.

The result follows from Lemmas 4, 5, and 6. \square

Lemma 4. *In the setting of Theorem 2, $PS(>_1, >_{-1}) \neq PS(>'_1, >_{-1})$ if and only if*

1. *there exists $k \geq m_a$ such that $\tau_1 \leq \dots \leq \tau_k < \min(\tau_x, \tau_y) \leq 1$ and*
2. *for all $l \in \{1, \dots, m_a\}$ there exists $l' \in \{1, \dots, k\}$ with $a_l = j_{l'}$.*

Proof of Lemma 4. Show “ \Rightarrow ” Choose k such that j_k is the last of the a_1, \dots, a_{m_a} to run out. Suppose, $\tau_y \leq \tau_k$. Agent 1 is busy consuming shares of other goods until time τ_k , regardless of the reported order of x and y . After τ_k agent 1 consumes shares of x until it is exhausted. Because y was already exhausted before τ_k , agent 1 gets no shares of y . Under report $>'_1$, it would finish consuming other goods at τ_k and find good y exhausted. Hence, it would begin consuming shares of x immediately, just as it did under report $>_1$. Thus, the order in which x and y are reported does not matter for the times at which it consumes goods x and y . Because $>_1$ and $>'_1$ only differ in the order of x and y , the remaining goods are also consumed in the same order and at the same times. Hence, agent 1's allocation does not change.

The case for $\tau_x \leq \tau_k$ is analogous.

Because PS is nonbossy (Kesten and Ekici (2012)), we know that if the switch from $>_1$ to $>'_1$ did not change the allocation for agent 1, it did not change the allocation at all.

Show “ \Leftarrow ” Suppose the last of the goods a_1, \dots, a_{m_a} to be exhausted is j_k , and $\tau_k < \tau_y \leq \tau_x$. Then agent 1 gets no shares of y . If it switches its report to $>'_1$, it will receive a non-trivial share of y , hence the allocation changes.

Now suppose the opposite, namely $\tau_y > \tau_x$. Agent 1 begins consumption of x at time τ_k and then turns to y at time τ_x . Thus agent 1 receives $\tau_x - \tau_k$ shares of x and $\tau_y - \tau_x$ shares of y . When it switches its report to $>'_1$, it will consume shares of y between τ_k and τ'_y . We need to show that $\tau'_y - \tau_k > \tau_y - \tau_x$. If $\tau'_y \geq \tau_y$, this is clear, because $\tau_k < \tau_x$ by assumption. In the following we assume $\tau'_y < \tau_y$.

Let $n_y(\tau)$ be the number of agents other than agent 1 consuming shares of y at time τ . n_y is integer-valued and increasing in τ , and there must exist a $\delta > 0$ such that $n_y(\tau_y - \delta) \geq 1$. This means that agent 1 is not the only agent consuming shares of y before it is exhausted. Otherwise, agent 1 would exhaust y alone, which implies that agent 1 received no shares of x , a contradiction.

If agent 1 reports $>'_1$ instead, let $n'_y(\tau)$ be the corresponding number of agents consuming y at times τ . We observe that x will be exhausted later, because agent 1 is no longer consuming shares of it. This means that agents who prefer x over y will arrive later at y . Agents arriving at y from other goods than x may also arrive later, because they face less competition from the agents stuck at x , etc. Therefore $n'_y \leq n_y$.

Under report $>_1$ from agent 1, y is exhausted by τ_y , i.e.,

$$q_y = \int_0^{\tau_y} n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_x\}} d\tau, \quad (21)$$

and under report $>'_1$, y is exhausted by τ'_y , i.e.,

$$q_y = \int_0^{\tau'_y} n'_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau \leq \int_0^{\tau'_y} n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau. \quad (22)$$

Equating (21) and (22) gives

$$\begin{aligned} \int_0^{\tau_y} n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_x\}} d\tau &\leq \int_0^{\tau'_y} n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau \\ \Rightarrow \int_{\tau'_y}^{\tau_y} n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau &\leq \int_0^{\tau'_y} \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau - \int_0^{\tau_y} \mathbb{1}_{\{\tau \geq \tau_x\}} d\tau + \int_{\tau'_y}^{\tau_y} \mathbb{1}_{\{\tau \geq \tau_k\}} d\tau \\ &= \int_0^{\tau_y} \mathbb{1}_{\{\tau \geq \tau_k\}} - \mathbb{1}_{\{\tau \geq \tau_x\}} d\tau \\ &= \tau_x - \tau_k. \end{aligned}$$

We know that j_k is exhausted before τ'_y and hence $n_y(\tau) + \mathbb{1}_{\{\tau \geq \tau_k\}} \geq 1$ for $\tau \in [\tau'_y, \tau_y]$, and ≥ 2 for $\tau \in [\tau_y - \delta, \tau_y]$. This yields

$$\tau_y - \tau'_y < \tau_x - \tau_k,$$

or equivalently $\tau_y - \tau_x < \tau'_y - \tau_k$. □

Lemma 5. *In the setting of Theorem 2, $RSD(>'_1, >_{-1}) \neq RSD(>_1, >_{-1})$ if and only if there exists a sequence (c_1, \dots, c_{k_c}) of k_c agents such that if RSD chose these agents first and in this order, they remove all goods a_1, \dots, a_{m_a} (and possibly more), but neither x , nor y .*

Proof of Lemma 5. In the *RSD* mechanism, a permutations of agents is chose amongst all possible permutations with uniform probability. The probability for agent 1 to get some good j is

$$P[1 \text{ gets } j] = \frac{|\{\pi \text{ permutation of } N : 1 \text{ gets } j \text{ under } \pi\}|}{|\{\pi \text{ permutation of } N\}|},$$

where the denominator is $n!$, and each permutation under which agent 1 gets j contributes $\frac{1}{n!}$ to the total probability.

For some permutation π consider the turn of agent 1. There are 5 possible cases:

1. Agent 1 faces a choice set including some a_l 's. This makes no contribution to its chances of getting x or y .
2. Agent 1 faces a choice set consisting only of b_l 's. Again, this makes no contribution to its chances of getting x or y .
3. Agent 1 faces only b_l 's and x , but not y . This case contributes $\frac{1}{n!}$ to its chances of getting x . This contribution is independent of the order in which it ranked x and y in its report.
4. Agent 1 faces only b_l 's and y , but not x . This case contributes $\frac{1}{n!}$ to its chances of getting y and the contribution is again independent of the ranking of x and y .
5. Agent 1 faces x , y and some b_l 's, but no a_l 's. This case contributes $\frac{1}{n!}$ to either the probabilities for x or y , depending on the ranking.

Show “ \Rightarrow ” If changing from $>_1$ to $>'_1$ influences the allocation, the allocation for agent 1 must have changed. This is because *RSD* is nonbossy. *RSD* also is SP, hence by Lemma 1 the probabilities for goods x and y must have changed. In all but the last case, the chances do not depend on the order in which x and y are reported. Thus, at least one permutation leads to case (5). This means that the sequence of agents chosen prior to agent 1 removes all a_l 's, but neither x nor y .

Show “ \Leftarrow ” Under report $>_1$, agent 1 will receive x any time case (5) occurs, while under $>'_1$ it will receive y . If a sequence (c_1, \dots, c_{k_c}) as defined in Lemma 5 exists, it is also the beginning of at least one permutation. When this permutation is selected, case (5) occurs. Switching from report $>_1$ to $>'_1$ thus strictly increases agent 1's chances of getting y .

□

Lemma 6. *In the setting of Theorem 2, (1) and (2) from Lemma 4 imply the existence of a sequence as described in Lemma 5.*

Proof of Lemma 6. We prove the claim by constructing a sequence of agents

$$(c_1, \dots, c_{k_c}) = (c_1^1, \dots, c_1^{q_1}, \dots, c_k^1, \dots, c_k^{q_k})$$

inductively. Under *RSD* this sequence will remove goods j_1, \dots, j_k in this order.

Selection of $c_k^1, \dots, c_k^{q_k}$ By assumption j_k was consumed strictly before x , hence $\tau_k < 1$. Then at least $q_k + 1$ agents receive non-trivial shares of j_k . Otherwise, if only q_k agents received shares of j_k , they would get the entire capacity and take time 1 to consume it, a contradiction. Select q_k of these agents other than agent 1 as $c_k^1, \dots, c_k^{q_k}$.

Because all $c_k^1, \dots, c_k^{q_k}$ actually received shares of j_k under PS , they must all prefer j_k to all other goods except for possibly j_1, \dots, j_{k-1} . In other words, suppose that j_1, \dots, j_{k-1} were removed under RSD in previous turns, the selected agents would remove j_k completely if chosen next (in arbitrary order).

Selection of $c_l^1, \dots, c_l^{q_l}, l < k$ Suppose, $c_{l+1}^1, \dots, c_k^{q_k}$ have been selected. Suppose further that m_l agents (plus possibly agent 1) receive non-trivial shares of j_l under PS . There are two cases:

Case 1: At least q_l of the m_l agents have not been selected as any of the $c_{l+1}^1, \dots, c_k^{q_k}$ so far. Then these agents are chosen as $c_l^1, \dots, c_l^{q_l}$.

Case 2: Only $n_l < q_l$ of the m_l agents have not been selected so far. The rest of the m_l agents have been selected at k' other goods. Let these goods be $j_{\rho(1)}, \dots, j_{\rho(k')}$ with $\rho(l') \in \{l+1, \dots, k\}$ for all $l' \in \{1, \dots, k'\}$. At each of the goods $j_{\rho(l')}$, $q_{\rho(l')}$ agents are selected. Now there must be at least $q_l - n_l + 1$ additional agents (possibly including agent 1) consuming non-trivial shares of the goods $j_{\rho(l')}$, otherwise at most $n_l + q_{\rho(1)} + \dots + q_{\rho(k')} + q_l - n_l$ agents fully consume goods $j_l, j_{\rho(1)}, \dots, j_{\rho(k')}$. This will take them until time 1, a contradiction.

There are two possible cases for these additional $q_l - n_l$ agents (excluding agent 1).

Case 2.1 All of them are available for selection. Then they are selected for the goods $j_{\rho(l')}$ of which they consume non-trivial shares, and the now free agents can be selected for j_l .

Case 2.1 Some of these agents are selected at some other goods $j_{\rho(k'+1)}, \dots, j_{\rho(k'+k'')}$. Then we use the free agents as in case 2.1, say $n_{l'}$. Then we still need $q_l - n_l - n_{l'}$ agents for j_l . There must be at least $q_l + q_{\rho(1)} + \dots + q_{\rho(k'')} + 1$ agents consuming non-trivial shares of the goods $j_l, j_{\rho(1)}, \dots, j_{\rho(k'')}$. $q_l - n_l - n_{l'}$ are not selected for any of these goods. Again there are two cases.

We repeat this argument inductively until enough agents are found who are still available and can replace agents such that the need at good j_l can be satisfied. This must happen, otherwise all agents selected so far as $c_{l+1}^1, \dots, c_k^{q_k}$, some $n_l''' < q_l$ agents and possibly agent 1 fully consume goods j_l, j_{l+1}, \dots, j_k goods, again a contradiction.

The fact that all selected agents $c_l^1, \dots, c_l^{q_l}, l \in \{1, \dots, k\}$ receive a non-trivial share in the goods j_l implies that they each prefer j_l to all other goods, except possibly j_1, \dots, j_{l-1} . Thus, the sequence $(c_1^1, \dots, c_k^{q_k})$ has the properties needed for 5. \square

The following Lemma shows that any Random Serial Dictatorship (independent of the distribution over orderings) is nonbossy.

Lemma 7. *For any distribution over orderings, the respective RSD mechanism is nonbossy.*

Proof. Fix a distribution over orderings of the agents and let p_π be the probability that ordering π is chosen. Suppose that RSD is bossy, then there exists an agents i, i' , types t_i, t'_i , and $t_{-i} \in T^{n-1}$ such that $f(t_i, t_{-i})_i = f(t'_i, t_{-i})_i$, but $f(t_i, t_{-i})_{i'} \neq f(t'_i, t_{-i})_{i'}$. For the sake of brevity, we write t and t' for t_i and t'_i , respectively.

Let $\text{Can}(t, t') = (t_0 = t, t_1, \dots, t_{k-1}, t_k = t')$ be the canonical transition from $t = t_i$ to $t' = t'_i$. As in the proof of Theorem 1, the fact that the allocation is the same at the start and at the end of the transition implies that the allocation never changes during the transition, i.e., $f(t_l)_i = f(t_{l+1})_i$ for all $l \in \{0, \dots, k-1\}$. Recall that under strategyproof mechanisms, the effect of swaps in the canonical transition is never undone by subsequent swaps and that swaps only effect the probabilities for adjacent goods (see Lemma 1).

But the allocation changed for agent i' , hence it must have changed for agent i' at some swap in the transition, say from $t_{l'}$ to $t_{l'+1} \in N_{t_{l'}}$. Let j', j'' be the goods that were swapped in this transition. Consider an ordering of the agents π with $p_\pi > 0$. There are two cases.

- Agent i gets the same good under $t_{l'}$ as under $t_{l'+1}$. Then the swap had no effect on the allocation of any other agent, i.e., under π the swap does not change the allocation of the other agents.
- Agent i receives j' under $t_{l'}$, but j'' under $t_{l'+1}$. Then the swap changes the allocation of the agent that received j'' under $t_{l'}$. The magnitude of the change is $-p_\pi < 0$. This agent can be i' by assumption.

However, the latter case is impossible, because this would also strictly increase agent i 's chances of receiving j'' (by $p_\pi > 0$), implying $f(t_{l'})_i \neq f(t_{l'+1})_i$, a contradiction. \square

E.2. Proofs from Section 8.2: Impossibility Result for Mixing RSD and RV

Proof of Proposition 11. If v is a rank valuation with non-decreasing increments, then for any number of goods $m \geq 3$ there exists a setting (N, M, \mathbf{q}) and preference profile $\succ = (\succ_1, \succ_{-1})$ such that agent 1 can beneficially manipulate RV, but RSD is invariant to this manipulation.

Suppose, $m \geq 3$ goods must be allocated and $v_k - v_{k+1} < v_{k+1} - v_{k+2}$, i.e., v has non-decreasing increments. Then consider a setting with $n = m$ agents and unit capacity for each good. Let \succ be the preference profile defined by

$$\begin{aligned} \succ_1 & : a_1 > \dots > a_{k-1} > b > c > d > e_{k+3} \dots > e_m, \\ \succ_i, i = 2, \dots, m & : a_1 > \dots > a_{k-1} > d > b > c > e_{k+3} \dots > e_m. \end{aligned}$$

Under RV and truthful reports, agent 1 will get good c with certainty. To see this suppose that agent 1 gets b instead. Then some other agent i received c . If agent 1 and agent i trade, the objective in (8) will decrease by $v_k - v_{k+1}$, because agent 1's suffers. However, agent i benefits from this trade, and social value also increases by $v_{k+1} - v_{k+2}$. By the non-decreasing assumption, this trade represents an improvement of the objective. Now suppose that agent 1

gets another good than b or c . Again some agent i gets good c . If agent 1 and agent i trade, this improves the objective, because the rank distribution improves (independently of v).

We have argued that agent 1 will get good c in any deterministic rank efficient allocation. But since rank efficient allocations are always convex combinations of deterministic rank efficient allocations (see Claim 8 and Theorem 1 in Featherstone (2011)), agent 1 must get good c with certainty.

Suppose now that agent 1 reports

$$>'_1 : a_1 > \dots > a_{k-1} > b > d > c > e_{k+3} \dots > e_m$$

instead, i.e., it swaps goods c and d in its report. Then under any rank efficient allocation (rank efficient with respect to $(>'_1, >_{-1})$), agent 1 will receive good b . This is because whenever agent 1 gets another good in some deterministic allocation, the objective improves if agent 1 trades with the agent who received b (independent of v). By the same argument as above, no rank efficient allocation will give agent 1 any other good than b . Thus, swapping c and d in its report is a beneficial manipulation for agent 1. This is independent of its actual utility, as long as the utility is consistent with $>_1$.

Now consider the outcome of the RSD mechanism. If $RSD(>_1, >_{-1})_1 \neq RSD(>'_1, >_{-1})_1$, we know from Lemma 1 that the only probabilities that change for agent 1 are those for c and d . Consider an ordering π of the agents.

- If $\pi(1) \leq k - 1$ (i.e., agent 1 is one of the first $k - 1$ agents in the ordering), it gets one of the goods $a_{\pi(1)}$.
- If $\pi(1) \in \{k, k + 1\}$, it gets good b .
- If $\pi(1) = k + 2$, it gets good c . This is independent of the order in which it reports its preferences over c and d , since d is already taken by some agent i with $\pi(i) = k$.
- If $\pi(1) \geq k + 3$, goods a_l, d, b, c have been consumed by other agents (in this order), hence agent 1 gets a good $e_{\pi(1)}$.

We see that regardless of the order in which agent 1 reports its preference over c and d , it will never receive d . Hence, its probability for d does not change under the misreport $>'_1$. This is a contradiction to $RSD(>_1, >_{-1})_1 \neq RSD(>'_1, >_{-1})_1$. \square

F. Examples

Example 5. Suppose a setting with 4 agents $\{1, 2, 3, 4\}$ and 4 goods $\{a, b, c, d\}$ with unit capacity. Agents' preferences are

$$\begin{aligned} >_1 & : a > d > \dots, \\ >_2 & : a > b > d > c, \\ >_3 & : b > c > d > a, \\ >_4 & : c > a > \dots \end{aligned}$$

The unique rank efficient allocation is given by $x_{1,d} = x_{2,a} = x_{3,b} = x_{4,c} = 1$. Suppose now that agent 1 reports

$$>'_1 : a >' b >' c >' d,$$

then the unique rank efficient allocation changes to $x_{1,a} = x_{2,d} = x_{3,b} = x_{4,c} = 1$. This means that agent 1 can manipulate to receive its first rather than its second choice. Note that this manipulation is beneficial for agent 1, regardless of its cardinal utility and of the actual rank efficient mechanism.

Remark 5. [Details for Examples 3 and 4] Consider the following ex-post efficient deterministic allocations:

$$\begin{aligned} A_{1,3,4,2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_{2,3,4,1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ A_{1,4,3,2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_{2,4,3,1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ A_{3,2,1,4} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{3,1,2,4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ A_{4,2,1,3} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_{4,1,2,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} B_{1,2,3,4} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_{3,2,4,1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ B_{2,1,4,3} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B_{4,1,3,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

In Example 3, Y_1 is a combination of all $A_{...}$ with equal weights. X_1 is a combination of all $B_{...}$ with equal weights. Y_2 is a combination of Y_1 and X_1 with equal weights. In Example 4, X_2 is a combination where the $A_{1,...}, A_{2,...}$ each have twice the weight of allocations $A_{3,...}, A_{4,...}$.

Example 6. Fix a setting (M, N, \mathbf{q}) and bounds (r, B) . Suppose, agent 1 is of type t and has utility $u \in t, u \notin \text{quasi-URB}(r, B)$. We now construct a strategyproof mechanism f and a manipulable mechanism g , such that $h_{\beta_{\max}}$ is strategyproof. Then it is also u -strategyproof. Thus, in this case the condition quasi-URB does not bind.

Consider mechanisms that react to the preferences of agent 1 and simply distributes the remaining shares of goods among the other agents. No other agent has any incentives, because their reports are not considered. Under f agent 1 receives probability $\frac{1}{2} + \epsilon$ for its highest ranking good, and $\frac{1}{2} - \epsilon$ for its second choice. This mechanism is obviously strategyproof and changes the allocation whenever agent 1 changes one of its two highest ranking goods. Let g be the mechanism that allocates $\frac{1}{2} - \epsilon$ and $\frac{1}{2} + \epsilon$ for the first and second choice, respectively. It is obviously manipulable and $g \ll f$.

In this case, $\beta_{\max} = \frac{1}{2}$, regardless of the bounds (r, B) , and $h_{\beta_{\max}}$ simply gives agent 1 probability $\frac{1}{2}$ for its two highest ranking choices. This mechanism is strategyproof, hence it is u -strategyproof.

G. Definitions

In the following definitions, we use the reduced expressions $h(t)$ to denote $h(t_i, t_{-i})_i$ and drop the index i .

Definition 11 (Adapted from Definition 6. in [Pathak and Sönmez \(2013\)](#)). Mechanism h' is as intensely and strongly manipulable as mechanism h if for any type t and utility $u \in t$, any misreport t' , any $t_{-i} \in T^{n-1}$, and any $\epsilon > 0$,

$$\langle u, h(t') - h(t) \rangle > 0 \Rightarrow \exists t^* \in T : \langle u, h'(t^*) - h'(t) \rangle > \langle u, h(t') - h(t) \rangle - \epsilon. \quad (23)$$

Definition 12 (Adapted from Definition 7. in [Pathak and Sönmez \(2013\)](#)). Mechanism h is intensely and strongly more manipulable than mechanism h' if

1. h' is as intensely and strongly manipulable as h , and
2. there exists $t \in T, u \in t, t_{-i} \in T^{n-1}$ such that
 - for all $t' \in T : \langle u, h(t) - h(t') \rangle \geq 0$, and
 - there exists $t^* \in T : \langle u, h'(t) - h'(t') \rangle < 0$.

H. Algorithms

ALGORITHM 2: Random Serial Dictatorship with Uniform Probability

Input: type report (t_1, \dots, t_n) , agents N , goods M , capacities \mathbf{q}

Variables: X allocation, interim capacities \mathbf{q}'

begin

```

 $X \leftarrow (x_{i,j} = 0)_{i \in N, j \in M}$ 
for  $\pi \in \text{Permutations}(N)$  do
  for  $i \in N$  do
     $j \leftarrow \text{BestAvailableGood}(t_{\pi(i)}, \mathbf{q}')$ 
     $x_{\pi(i),j} \leftarrow x_{\pi(i),j} + 1$ 
     $q'_j \leftarrow q'_j - 1$ 
  end
end
 $X \leftarrow \frac{1}{n!} X$ 

```

end

ALGORITHM 3: Probabilistic Serial with Uniform Eating Speeds

Input: type report (t_1, \dots, t_n) , agents N , goods M , capacities \mathbf{q}

Variables: X allocation, interim capacities \mathbf{q}' , τ_1, \dots, τ_m run-out times

begin

```

 $X \leftarrow (x_{i,j} = 0)_{i \in N, j \in M}$ 
 $k \leftarrow 1, \tau \leftarrow 0$ 
while  $\tau < 1$  do
  for  $j \in M$  do
     $n_j \leftarrow \text{numberAgentsForWhomJBestOfAvailableGoods}(j, t_1, \dots, t_n, \mathbf{q}')$ 
  end
   $\Delta\tau_k \leftarrow \min\left(\min_{j \in M} \left(\frac{q'_j}{n_j}\right), 1 - \tau\right)$ 
   $\tau \leftarrow \tau + \Delta\tau_k$ 
  for  $i \in N$  do
     $j \leftarrow \text{BestAvailableGood}(t_i, \mathbf{q}')$ 
     $x_{i,j} \leftarrow x_{i,j} + \Delta\tau_k$ 
     $q'_j \leftarrow q'_j - \Delta\tau_k$ 
  end
   $k \leftarrow k + 1$ 
end

```

end
